

# D-FINITE SYMMETRIC FUNCTIONS.

by

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# Abstract

The fruitful relation between the theory of symmetric functions and that of D-finite power series was first introduced by Goulden and Jackson in 1980, and later extended by Gessel, who stated two important results that provide closure properties of D-finite symmetric series under the scalar and inner products. These products are very important from the computational and combinatorial points of view, as a prime tool for coefficient extraction in symmetric series. Gessel presented some enumerative problems that can be better understood using his results on D-finiteness. We connect these notions with Scharf, Thibon and Wybourne's results on reduced Kronecker products. Also, we extend the necessary conditions on one of Gessel's theorems and determine some consequences in Young Tableaux enumeration.

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# Contents

<b>Approval</b>	<b>ii</b>
<b>Abstract</b>	<b>iii</b>
<b>Acknowledgments</b>	<b>iv</b>
<b>Contents</b>	<b>v</b>
<b>List of Tables</b>	<b>vii</b>
<b>List of Figures</b>	<b>viii</b>
<b>1 Introduction</b>	<b>1</b>
1.1 D-finite symmetric functions . . . . .	1
1.2 Computations and applications of D-finite symmetric functions . . . . .	2
1.3 Reduced Kronecker product . . . . .	2
1.4 An outline . . . . .	3
<b>2 Generating Functions</b>	<b>4</b>
2.1 Basics . . . . .	4
2.2 Classes of generating functions . . . . .	6
<b>3 Symmetric functions</b>	<b>8</b>
3.1 Partitions . . . . .	8
3.2 Some important symmetric functions . . . . .	11
3.3 Relations between the bases of $\Lambda$ . . . . .	13
3.4 Schur functions . . . . .	16

3.5	Operations on symmetric functions . . . . .	17
3.5.1	Plethysm . . . . .	17
3.5.2	Scalar product of Symmetric functions . . . . .	19
3.5.3	Application to Schur functions . . . . .	20
3.5.4	Kronecker product of symmetric functions . . . . .	21
<b>4</b>	<b>D-finite functions</b>	<b>22</b>
4.1	Definitions . . . . .	22
4.2	Closure Properties . . . . .	26
4.3	Symmetric Series . . . . .	28
4.4	Plethysm . . . . .	29
4.5	Schur Functions . . . . .	30
<b>5</b>	<b>D-finiteness: scalar and Kronecker products</b>	<b>34</b>
5.1	Kronecker product . . . . .	34
5.1.1	Some Kronecker product identities . . . . .	35
5.2	Scalar product . . . . .	39
5.2.1	Extension of closure properties for D-finite scalar products . . . . .	41
5.3	Applications to tableaux enumeration . . . . .	42
5.4	Other applications . . . . .	44
<b>6</b>	<b>Reduced Kronecker product</b>	<b>47</b>
6.1	Reduced Kronecker coefficients . . . . .	47
6.2	Adjoint multiplication . . . . .	50
6.3	Applying the adjoint multiplication . . . . .	52
6.4	Relation to D-finiteness . . . . .	54
<b>7</b>	<b>Conclusion</b>	<b>58</b>
7.1	Summary . . . . .	58
7.2	Open problems . . . . .	59
	<b>Bibliography</b>	<b>61</b>

# List of Tables

5.1	A family of Kronecker product identities . . . . .	38
5.2	A family of scalar product identities. . . . .	42
5.3	Classes of labelled graphs. . . . .	45
6.1	Coefficients for two-row shape Schur functions . . . . .	50

# List of Figures

3.1	Young diagram of shape $\lambda = (7, 6, 4, 3, 3, 2, 1, 1, 1)$ and its conjugate. . . . .	9
3.2	SSYT of shape $\lambda = (7, 6, 4, 3, 3, 2, 1, 1, 1)$ and type $(2, 3, 3, 4, 5, 3, 2, 2, 1, 0, 1)$ . . . . .	10
3.3	SSYT of skew shape $\lambda/\mu = (7, 6, 4, 3, 3, 2, 1, 1, 1)/(3, 3, 1, 1)$ . . . . .	10



# Chapter 1

## Introduction

Symmetric functions are classic mathematical objects which serve as powerful devices for establishing deep relations between areas of mathematics. These are simply multivariate power series that are invariant under any permutation of their variables. The Kronecker product (also referred to as inner product) and the scalar product, which can be defined by bilinearity from their definition on some of the five widely known bases  $\{p_\lambda\}$ ,  $\{m_\lambda\}$ ,  $\{h_\lambda\}$ ,  $\{e_\lambda\}$  and  $\{s_\lambda\}$  of the space  $\Lambda$  of symmetric functions, are of special importance in the theory of representations of symmetric groups, and are also helpful in the resolution of some combinatorial problems involving coefficient extraction from symmetric series. There are numerous open problems involving these products and their study is a very active topic in algebraic combinatorics. Our interest here is mostly related to enumerative applications.

### 1.1 D-finite symmetric functions

Information on the coefficients of a power series may sometimes be translated into the solution to an enumerative problem. Gessel [7] applied this idea to symmetric functions, where coefficient extraction is achieved through the use of the scalar and inner products. In further sections we present some of the enumerative problems tackled by Gessel. In some cases it may not be possible to obtain explicit formulas for the coefficients of a given symmetric function, but we can get some of its analytic properties. For this purpose, Gessel resorted to the theory of D-finite formal series. A power series in one variable is D-finite if it satisfies a linear homogenous differential equation with polynomial coefficients. It is possible to extend this concept to more than one variable and also to an infinite number of variables.

Gessel considers symmetric functions as objects that exist as a series in  $\mathbb{Q}[[p_1, p_2, \dots, p_n, \dots]]$ , and defined a symmetric function to be D-finite whenever it is D-finite with respect to the  $p_i$ 's, viewing each  $p_i$  as a formal variable. The importance of D-finiteness in enumeration comes from the fact that a generating function in one variable is D-finite if and only if its coefficients are P-recursive, i.e., they satisfy a linear recursion with polynomial coefficients.

Gessel [7] showed sufficient conditions for a scalar product and an inner product of symmetric functions to be D-finite. We state his results briefly here:

1. If  $f$  and  $g$  are symmetric functions that are D-finite in the  $p_i$ 's (and maybe in some other variable  $t$ ), then  $f * g$  is D-finite in these variables.
2. If  $f$  and  $g$  are D-finite with respect to the  $p_i$ 's and another variable  $t$ ,  $g$  involves only a finite number of  $p_i$ 's, and  $\langle f, g \rangle$  is well-defined as a formal power series in  $t$ , then  $\langle f, g \rangle$  is D-finite with respect to  $t$ .

One of our main motivations is to find weaker sufficient conditions for the second part. We show some fairly general families of symmetric functions that do not satisfy these conditions and whose scalar product is still D-finite.

## 1.2 Computations and applications of D-finite symmetric functions

Mishna [15] and Chyzak, Mishna and Salvy [5] developed algorithms that compute a system of differential equations satisfied by the Kronecker product of two symmetric functions given the differential equations satisfied by each of these functions. In [16], Mishna introduced a family of Kronecker product identities that were obtained using a Maple package that implements these algorithms, and a multiplicativity property of the Kronecker product. We use the same methods in order to obtain a family of D-finite scalar products that does not satisfy Gessel's conditions. These identities may suggest some ideas concerning the desired weaker conditions for a scalar product of symmetric functions to be D-finite.

## 1.3 Reduced Kronecker product

The coefficients that arise from expanding (in terms of Schur functions) the regular multiplication of Schur functions are known as the Littlewood-Richardson coefficients, and they

can be interpreted combinatorially by applying the well known Littlewood-Richardson rule. On the other hand, the coefficients that arise from the Kronecker product of Schur functions, which are called Kronecker coefficients (and can be proven to be positive integer using representation theory), have remained a mystery for many years, as no one has been able to find a general combinatorial interpretation or a closed formula for them. However, there have been some recent research on the reduced Kronecker product, which was introduced by Murnaghan in 1938, producing the reduced Kronecker coefficients, which may shed some light on the nature of the Kronecker coefficients. Murnaghan's work focuses mainly on the reduced Kronecker product of Schur functions. Thibon [24] exploited the bialgebra and Hopf algebra structure given by the Kronecker product and reduced Kronecker product of symmetric functions in order to extend Murnaghan's work and obtain very general Kronecker product identities. We do not develop Thibon's theory here, but we present his results and show that an important particular case of his main theorem can be obtained using the algorithms from [15] and [5]. We conjecture that a more general case could be deduced using these algorithms.

We also show how Thibon's results may help us obtain an extension of Gessel's theorem on the closure properties of D-finiteness under the scalar product.

## 1.4 An outline

We recollect in Chapters 2 and 3 the basic concepts on generating functions and symmetric functions. Most of the notation we use is based on Stanley [22] and Macdonald [14]. Also, we recall the notion of D-finiteness in Chapter 4 along with the extension to symmetric functions. In Chapter 5 we state Gessel's closure properties of D-finiteness with respect to the scalar and Kronecker products. Then we present an extension of it in Chapter 6 together with the reduced notation of symmetric functions and some of Thibon's results. Also, we explain one of his results from the D-finiteness point of view and state some open problems for future research.

## Chapter 2

# Generating Functions

Generating functions play an essential role in combinatorics, by encoding counting information of combinatorial classes. They are of special importance in the study of operations between classes as is the case in the theory of combinatorial structures. They are also useful in the absence of a general closed formula for the number of objects of a given size in a combinatorial class.

### 2.1 Basics

A combinatorial *class* is a pair  $(\mathcal{C}, |\cdot|)$  where  $\mathcal{C}$  is a finite or denumerable set and the mapping  $|\cdot| : \mathcal{C} \mapsto \mathbb{Z}_{\geq 0}$  assigns to every element  $c \in \mathcal{C}$  a nonnegative integer called the *size of c*, denoted by  $|c|$ , such that the number of elements of any given size is finite. It becomes natural then to consider the sequence  $(C_n)_{n \geq 0}$ , where  $C_n$  denotes the number of elements of size  $n$  in  $\mathcal{C}$ .

A *formal power series* is defined as an infinite formal sum of monomials in some set of variables. The notion of convergence are not of foremost importance in our current work. Formal power series are defined over a ring of coefficients. We are going to be working mainly on the ring  $\mathbb{Q}$  of rational numbers. In the univariate case, power series are added and multiplied in a natural way as follows:

$$\left( \sum_{n \geq 0} f_n z^n \right) + \left( \sum_{n \geq 0} g_n z^n \right) = \sum_{n \geq 0} (f_n + g_n) z^n.$$

$$\left( \sum_{n \geq 0} f_n z^n \right) \times \left( \sum_{n \geq 0} g_n z^n \right) = \sum_{n \geq 0} \left( \sum_{k=0}^n f_k g_{n-k} \right) z^n.$$

The coefficient of  $z^n$  in a formal power series  $f(z)$  is denoted by  $[z^n]f(z)$ .

*Generating functions* are particular cases of formal power series that depend on combinatorial classes. Given a class  $\mathcal{C}$ , its *ordinary generating function* is defined by:

$$C(z) = \sum_{n \geq 0} C_n z^n,$$

where  $C_n$  is the number of elements of size  $n$  in  $\mathcal{C}$ . Also define the *exponential generating function* of the class  $\mathcal{C}$  as the formal power series

$$\tilde{C}(z) = \sum_{n \geq 0} C_n \frac{z^n}{n!}.$$

Ordinary generating functions are normally used for classes of unlabelled objects while exponential generating functions are used to count classes of labelled objects.

**Example 2.1.** Denote by  $\mathcal{P}$  the class of all the permutations of sets of the form  $\{1, 2, \dots, n\}$ , where the size of a permutation  $\sigma : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$  is  $n$ . The number of objects of size  $n$  is  $P_n = n(n-1) \cdots 1 = n!$ , since a permutation of size  $n$  is a bijection of the set  $\{1, \dots, n\}$  into itself. There are  $n$  possibilities for the image of 1, then  $n-1$  for the image of 2 and so on. Thus we get the sequence  $(P_n)_{n \geq 0} = 1, 1, 2, 6, \dots, n!, \dots$ . The ordinary and exponential generating functions of  $\mathcal{P}$  are given respectively by:

$$P(z) = \sum_{n \geq 0} n! z^n,$$

$$\tilde{P}(z) = \sum_{n \geq 0} n! \frac{z^n}{n!} = \sum_{n \geq 0} z^n = \frac{1}{1-z}.$$

Notice that the ordinary generating function does not have a simple expression, while the exponential one does, which suggests it may be the more relevant series to study.

Multivariate formal power series are also important in combinatorics and serve to keep track of additional parameters. Consider a class  $(\mathcal{C}, |\cdot|)$  and a family of parameters  $\chi_i : \mathcal{C} \rightarrow \mathbb{Z}_{\geq 0}$ ,  $i = 1, \dots, d$ . Let  $\mathbf{u} = (u_1, u_2, \dots, u_d)$  be an ordered set of formal variables and for any  $\mathbf{k} = (k_1, k_2, \dots, k_d) \in \mathbb{N}^d$  denote  $\mathbf{u}^{\mathbf{k}} = u_1^{k_1} \cdots u_d^{k_d}$ . A *multivariate generating function*

(MGF), either ordinary or exponential, of the combinatorial class  $\mathcal{C}$ , is a formal power series in two or more variables  $z, u_1, u_2, \dots, u_d$ , defined by

$$\begin{cases} f(z, \mathbf{u}) = \sum_{n, \mathbf{k}} f_{n, \mathbf{k}} \mathbf{u}^{\mathbf{k}} z^n, & \text{(ordinary BGF),} \\ \tilde{f}(z, \mathbf{u}) = \sum_{n, \mathbf{k}} f_{n, \mathbf{k}} \mathbf{u}^{\mathbf{k}} \frac{z^n}{n!}, & \text{(exponential BGF).} \end{cases}$$

where  $f_{n, \mathbf{k}}$  is the multi-index sequence of the number of objects  $\varphi$  in the class  $\mathcal{C}$ , such that  $|\varphi| = n$  and  $\chi_j(\varphi) = k_j$  for  $1 \leq j \leq d$ .

## 2.2 Classes of generating functions

In this section we define some useful classes of generating functions in enumerative combinatorics. Namely the rational, algebraic and differentially finite (or D-finite) generating functions. The importance of these categories of generating functions is the fact that they all imply the existence of a relatively simple recurrence relation on their coefficients. A generating function, or more generally a formal power series,  $F(x)$  is *rational* if there exist polynomials  $r(x), t(x) \in \mathbb{Q}[x]$  such that

$$F(x) = \frac{r(x)}{t(x)}, \quad (2.1)$$

where  $t(0) \neq 0$ . As a simple example consider  $F(x) = \sum_{n \geq 1} x^n = \frac{x}{1-x}$ , which is a rational function.

A formal power series  $F \in \mathbb{Q}[[x]]$  is *algebraic* if

$$t_k(x)F(x)^k + t_{k-1}(x)F(x)^{k-1} + \dots + t_0(x) = 0, \quad (2.2)$$

for some polynomials  $t_0(x), \dots, t_k(x) \in \mathbb{Q}[x]$  with  $t_k \neq 0$ . The *degree* of  $F(x)$  is the smallest positive integer  $k$  for which an equation of the form (2.2) is satisfied. An example of an algebraic power series is

$$F(x) = \frac{1}{\sqrt{1-x}} = \sum_{n \geq 0} (-1)^n \binom{-1/2}{n} x^n,$$

where  $-1 + (1-x)F(x)^2 = 0$ , so the polynomials corresponding to equation 2.2 are  $t_0(x) = -1$ ,  $t_1(x) = 0$ ,  $t_2(x) = 1-x$  and  $t_i(x) = 0$  for  $i > 2$ . Notice that  $F(x)$  has degree 2.

An example of a formal power series that is not algebraic is the following:

$$F(x) = \exp(x) = \sum_{n \geq 0} \frac{x^n}{n!}.$$

To prove that  $\exp(x)$  is not algebraic, suppose that it is, and let  $d$  be the smallest degree of a polynomial equation

$$a_0 + a_1 \exp(x) + \cdots + a_d \exp(dx) = 0,$$

satisfied by  $\exp(x)$ . Differentiating this polynomial equation we obtain another one

$$a_1 \exp(x) + \cdots + da_d \exp(dx) = 0,$$

and by subtracting the first equation times  $d$  to the second one, we obtain one with degree smaller than  $d$ , which is a contradiction.

It is easy to see that any rational power series is also an algebraic power series. It is well known that the coefficients of both rational and algebraic generating functions satisfy simple recursions (Stanley [22]). The class of D-finite functions was introduced in 1980 by Stanley [21], given by generating functions whose coefficients satisfy more general recursions. A generating function is differentially finite if all of its derivatives span a finite-dimensional vector space. We give a more formal definition of differentially finite generating functions in Section 4.1. This formal definition easily extends to multiple variables, an infinite number of variables and to the notion of D-finite symmetric functions.

## Chapter 3

# Symmetric functions

As stated in the introduction, we are mainly interested in improving our understanding of the scalar and inner products of symmetric functions. We define a *symmetric function* on  $x_1, x_2, \dots, x_n, \dots$  as a formal power series that is invariant under any (possibly infinite) permutation of variables. Let  $\mathbb{Q}$  be a field of characteristic zero and consider the *space of symmetric functions of degree  $n$  over  $\mathbb{Q}$* , denoted by  $\Lambda^n$ . If we take the direct sum of the  $\Lambda^i$ 's, i.e:  $\Lambda = \bigoplus_{i \geq 0} \Lambda^i$ , we obtain the space of all the symmetric functions, which is also a graded algebra.

### 3.1 Partitions

Both the inner and scalar products of symmetric functions are often defined by setting the value of these operations on the relevant bases of  $\Lambda$  given in Section 3.2. All of these bases are indexed by integer partitions, so let us first recall some elementary concepts concerning partitions. A *partition*  $\lambda = (\lambda_1, \dots, \lambda_k)$  of a nonnegative integer  $n$  is a non-increasing sequence of integers  $\lambda_i$  (*parts*) such that  $\sum_i \lambda_i = n$ . We write  $\lambda \vdash n$  and  $|\lambda| = n$  and we say that  $n$  is the *size* of  $\lambda$ . The empty partition is denoted by  $(0) \vdash 0$  and is assumed to have zero parts. Other important parameters of the partition  $\lambda$  are denoted by  $l(\lambda)$  and  $m_i(\lambda)$  (or simply  $m_i$ ), where  $l(\lambda)$  is known as the *length* of  $\lambda$  and counts the number of nonzero parts of  $\lambda$ , while  $m_i$  denotes the number of parts of  $\lambda$  that are equal to  $i$ . Using this last parameter, we can write a partition using another notation  $\lambda = 1^{m_1} 2^{m_2} \dots r^{m_r}$ , which is frequently convenient. Let us now define the integers  $z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \dots r^{m_r} m_r!$  and  $\varepsilon_\lambda = (-1)^{m_2 + m_4 + \dots}$ . Also set  $\mathcal{P} := \bigcup_{n \geq 0} \mathcal{P}(n)$  with  $\mathcal{P}(n)$  denoting the set of all partitions



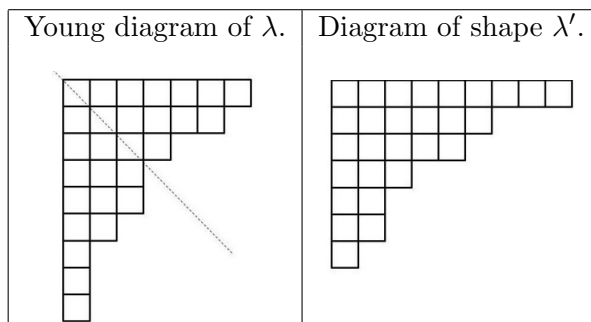


Figure 3.1: Young diagram of shape  $\lambda = (7, 6, 4, 3, 3, 2, 1, 1, 1)$  and its conjugate.

of  $n$ .

A partition can be visually represented as a Young diagram. A *Young diagram* is a finite collection of boxes into rows, where the number of boxes in each row is greater than or equal to the number of boxes in the next one below. Each part of a partition is equal to the number of boxes in the corresponding row of its Young diagram. Notice that this representation is well-defined and unique. Define now the *conjugate*, also known as transpose, of a partition  $\lambda$ , as the partition obtained by reflecting the Young diagram of shape  $\lambda$  along its main diagonal. We denote it by  $\lambda'$ .

Figure 3.1 shows the Young diagrams of the partition  $\lambda = (7, 6, 4, 3, 3, 2, 1, 1, 1) = 1^3 2^1 3^2 4^1 6^1 7^1$  of size 28 and length 9 and its conjugate  $\lambda' = (9, 6, 5, 3, 2, 2, 1)$ .

A *standard Young tableau* (SYT) of size  $n$  is a Young diagram of a partition  $\lambda \vdash n$  combined with a labelling of each box by the numbers 1 to  $n$  such that these integers are strictly increasing both in every row (from left to right) and in every column (from top to bottom). If we relax the strictly increasing condition in the rows so that these are weakly increasing, and we allow the labels to be positive integers without additional restrictions, we obtain what is called a *semi-standard Young tableau* (SSYT). The partition  $\lambda$  is called the *shape* of the SSYT (or SYT), and the labels of the boxes are its *entries*. Also, we say that a SSYT  $\mathbf{T}$  has *type*  $\alpha = (\alpha_1, \alpha_2, \dots)$  if  $\mathbf{T}$  has exactly  $\alpha_i$  entries equal to  $i$ . Notice that a SYT is simply a SSYT of type  $(1, 1, \dots, 1)$ .

**Example 3.1.** Consider the same partition  $\lambda = (7, 6, 4, 3, 3, 2, 1, 1, 1)$  from Figure 3.1. A SSYT of shape  $\lambda$  is illustrated in Figure 3.2.

Given two partitions  $\lambda$  and  $\mu$  such that  $\mu \subseteq \lambda$  (this is  $\mu_i \leq \lambda_i$  for all  $i$ ) define a *skew shape*

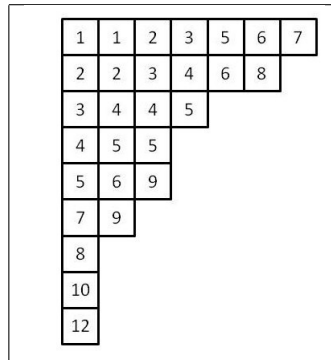


Figure 3.2: SSYT of shape  $\lambda = (7, 6, 4, 3, 3, 2, 1, 1, 1)$  and type  $(2, 3, 3, 4, 5, 3, 2, 2, 2, 1, 0, 1)$ .

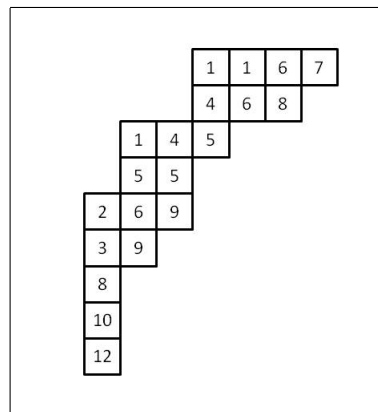


Figure 3.3: SSYT of skew shape  $\lambda/\mu = (7, 6, 4, 3, 3, 2, 1, 1, 1)/(3, 3, 1, 1)$

as the diagram  $\lambda/\mu$  obtained after deleting the cells corresponding to the Young diagram of  $\mu$ , from the Young diagram of  $\lambda$ . A *SSYT of skew shape*  $\lambda/\mu$  is defined the same way as a SSYT of a non-skew shape. See Figure 3.3 for example.

Finally recall that a *composition*  $\alpha \models n$  of a nonnegative integer  $n$  is simply a finite sequence  $\alpha = (\alpha_1, \dots, \alpha_k)$  of nonnegative integers that add up to  $n$ . A composition is said to be *weak* if at least one of its parts is equal to 0.

### 3.2 Some important symmetric functions

In this subsection we present some of the most important families of symmetric functions. The first relevant family is that of *monomial symmetric functions*, which are denoted by  $m_\lambda(\mathbf{x})$  (where  $\mathbf{x} = (x_1, x_2, x_3, \dots)$  and  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$  is a partition), and are given by:

$$m_\lambda(\mathbf{x}) := \sum_{\substack{i_1, i_2, \dots, i_k \geq 1 \\ \text{all different}}} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \cdots x_{i_k}^{\lambda_k} \text{ with } m_{(0)} = 1.$$

The second family consists of the *power sum symmetric functions* which are given by:

$$p_n(\mathbf{x}) := \sum_{i \geq 1} x_i^n \text{ for } n \geq 1 \text{ and } p_0 = 1;$$

$$p_\lambda(\mathbf{x}) := p_{\lambda_1} \cdots p_{\lambda_k}.$$

The *elementary symmetric functions*  $e_\lambda$  are defined as the sum:

$$e_n(\mathbf{x}) := \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n} \text{ for } n \geq 1 \text{ and } e_0 = 1;$$

$$e_\lambda(\mathbf{x}) := e_{\lambda_1} \cdots e_{\lambda_k}.$$

Now we define the *complete homogeneous symmetric functions*  $h_\lambda$  by the formulas:

$$h_n(\mathbf{x}) := \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n} \text{ for } n \geq 1 \text{ and } h_0 = 1;$$

$$h_\lambda(\mathbf{x}) := h_{\lambda_1} \cdots h_{\lambda_k}.$$

Notice that the complete homogeneous symmetric function indexed by  $n$  is the sum of all possible monomials of degree  $n$ , while the elementary are the sum of only the monomials of degree  $n$  where each variable does not appear more than once. In other words:

$$h_n = \sum_{\lambda \vdash n} m_\lambda \text{ and } e_n = m_{1^n}.$$

Last, but not least important, are the Schur symmetric functions, which can be defined as follows:

$$s_\lambda(\mathbf{x}) := \sum_{\mathbf{T}} \mathbf{x}^{\mathbf{T}}.$$

where the sum is over all SSYT of shape  $\lambda$ , and  $\mathbf{x}^{\mathbf{T}}$  denotes the monomial  $x_1^{r_1}x_2^{r_2}\cdots$ , where  $r_i$  is the number of times the entry  $i$  appears in  $\mathbf{T}$ . For instance, for the SSYT of Figure 3.1 we have:

$$\mathbf{x}^{\mathbf{T}} = x_1^2x_2^3x_3^3x_4^4x_5^5x_6^3x_7^2x_8^2x_9^2x_{10}x_{12},$$

which would be one term in the expansion of  $s_{(7,6,4,3,3,2,1,1,1)}(\mathbf{x})$ .

Notice that the monomial symmetric functions and the Schur functions do not satisfy the nice multiplicative property that defines the elementary, power sum and complete homogeneous symmetric functions. We will study Schur functions more deeply in Section 3.4.

**Example 3.2.** *Some examples of the symmetric functions defined above are the following:*

- $m_{(0)} = p_0 = h_0 = e_0 = s_{(0)} = 1.$
- $m_1 = \sum_i x_i,$   
 $m_{(2,1)} = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + \cdots = \sum_{i \neq j} x_i^2x_j.$
- $p_2 = x_1^2 + x_2^2 + x_3^2 + \cdots = \sum_i x_i^2.$   
 $p_{(2,1)} = p_2p_1 = x_1^3 + x_1^2x_2 + x_1^2x_3 + \cdots + x_2^3x_1 + x_2^3 + x_2^2x_3 + \cdots = \sum_i x_i^3 + \sum_{i \neq j} x_i^2x_j$   
 $= m_{(3)} + m_{(2,1)}.$
- $e_2 = x_1x_2 + x_1x_3 + x_1x_4 + \cdots = \sum_{i < j} x_ix_j.$   
 $e_{(2,1)} = e_2e_1 = x_1^2 + x_1x_2^2 + x_1x_2x_3 + x_1^2x_3 + x_1x_2x_3 + \cdots = m_{(2,1)} + 3m_{(1,1,1)}.$
- $h_2 = x_1^2 + x_1x_2 + x_1x_3 + x_1x_4 + \cdots + x_2^2 + x_2x_3 + x_2x_4 \cdots = m_{(1,1)} + m_{(2)}.$   
 $h_{(2,1)} = h_2h_1 = x_1^3 + x_1^2x_2 + x_1^2x_3 + \cdots + x_1^2x_2 + x_1x_2^2 + x_1x_2x_3 + \cdots + x_1^2x_3 + x_1x_2x_3 + x_1x_3^2 \cdots = 3m_{(1,1,1)} + 2m_{(2,1)} + m_{(3)}.$
- $p_n(\mathbf{x}) = m_n(\mathbf{x}).$

As suggested by the previous examples, it is always possible to write the symmetric functions from each of these families in terms of the others. In fact, the sets  $\{m_\lambda\}_{\lambda \vdash n}$ ,  $\{e_\lambda\}_{\lambda \vdash n}$ ,  $\{h_\lambda\}_{\lambda \vdash n}$ ,  $\{p_\lambda\}_{\lambda \vdash n}$  and  $\{s_\lambda\}_{\lambda \vdash n}$  are all linear bases of  $\Lambda^n$  over  $\mathbb{Q}^n$ .

**Theorem 3.1** (Symmetric functions fundamental theorem [22]). *Every set  $\{m_\lambda : \lambda \vdash n\}$ ,  $\{e_\lambda : \lambda \vdash n\}$ ,  $\{h_\lambda : \lambda \vdash n\}$ ,  $\{p_\lambda : \lambda \vdash n\}$ ,  $\{s_\lambda : \lambda \vdash n\}$  forms a linear basis for  $\Lambda^n$ .*

The complete proof of this theorem is omitted, but it can be found in [22].

**Corollary 3.2** ([22]). *Every set  $\{m_\lambda : \lambda \in \mathcal{P}\}$ ,  $\{e_\lambda : \lambda \in \mathcal{P}\}$ ,  $\{h_\lambda : \lambda \in \mathcal{P}\}$ ,  $\{p_\lambda : \lambda \in \mathcal{P}\}$ ,  $\{s_\lambda : \lambda \in \mathcal{P}\}$ , where  $\mathcal{P}$  is the set of all partitions, forms a linear basis for  $\Lambda$ .*

### 3.3 Relations between the bases of $\Lambda$

In this section we show the relations between power sum, elementary and complete symmetric functions with monomial symmetric functions among other relations between them.

Denote by  $M_{\lambda\mu}$  (where  $\lambda, \mu \models n$ ) the number of matrices  $A = (a_{ij})_{i,j}$  where  $a_{ij} \in \{0, 1\}$  such that the sum of the entries of the rows of  $A$  is  $\lambda$  ( $row(A) = \lambda$ ) and the sum of the entries of the columns of  $A$  is  $\mu$  ( $col(A) = \mu$ ). Notice that we may assume these matrices to be infinite, since we can always write any finite matrix as an infinite one by completing it with zeros.

**Example 3.3.** Set  $\lambda = (4, 3, 3, 2)$  and  $\mu = (4, 2, 2, 3, 1)$ . A matrix satisfying  $row(A) = (4, 3, 3, 2)$  and  $col(A) = (4, 2, 2, 3, 1)$  is given by:

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Now denote by  $N_{\lambda\mu}$  the number of matrices  $A = (a_{ij})_{i,j}$  where  $a_{ij} \in \mathbb{N}$  such that  $row(A) = \lambda$  and  $col(A) = \mu$  and by  $R_{\lambda\mu}$  the number of ordered set partitions  $\pi = (B_1, \dots, B_k)$  of the set  $[l(\lambda)] = \{1, \dots, l(\lambda)\}$  such that  $\mu_j = \sum_{i \in B_j} \lambda_i$ .

**Proposition 3.1** (see Stanley [22]).

1. The coefficient of  $\mathbf{x}^\alpha$  in  $e_\lambda$ , where  $\alpha$  is a weak composition and  $\lambda$  is a partition, is equal to  $M_{\lambda\alpha}$ , i.e:

$$e_\lambda(\mathbf{x}) = \sum_{\mu \vdash n=|\lambda|} M_{\lambda\mu} m_\mu(\mathbf{x}).$$

2. The coefficient of  $\mathbf{x}^\alpha$  in  $h_\lambda$ , where  $\alpha$  and  $\lambda$  are given as before, is equal to  $N_{\lambda\alpha}$ , this is:

$$h_\lambda(\mathbf{x}) = \sum_{\mu \vdash n} N_{\lambda\mu} m_\mu(\mathbf{x}).$$

3. The coefficient of  $\mathbf{x}^\mu$  in  $p_\lambda$  for  $\mu$  a partition is equal to  $R_{\lambda\mu}$ , i.e.:

$$p_\lambda(\mathbf{x}) = \sum_{\mu \vdash n} R_{\lambda\mu} m_\mu(\mathbf{x}).$$

*Proof.* (Sketch) We are going to present the idea for the first part of the proposition, as the other ones follow from the same type of argument. The idea of the proof is to find a bijection between the  $(0, 1)$ -matrices  $A = (a_{ij})_{i,j}$  with  $\text{row}(A) = \lambda$  and  $\text{col}(A) = \alpha$  and the monomials  $\mathbf{x}^\alpha$  that appear in the expansion of  $e_\lambda$ . Given a  $(0, 1)$ -matrix, the condition  $\text{row}(a) = \lambda$  means that we are choosing  $\lambda_1$  entries from the first row,  $\lambda_2$  from the second and so on. This is equivalent to selecting a term from  $e_{\lambda_1}$ , one from  $e_{\lambda_2}$ , etc. If we multiply all of them, we get a term of  $e_\lambda$ . On other hand, the condition  $\text{col}(A) = \alpha$  means that we are taking  $x_1$  exactly  $\alpha_1$  times,  $x_2$  exactly  $\alpha_2$  times, etc. In other words, the product is exactly  $\mathbf{x}^\alpha$ . Conversely, any monomial in  $e_\lambda$  corresponds to such matrix (see [22] for a proof of this correspondence). We have proven that every  $(0, 1)$ -matrix with these conditions corresponds to an occurrence of the term  $\mathbf{x}^\alpha$  in the expansion of  $e_\lambda$ . Therefore, the coefficient of  $\mathbf{x}^\alpha$  is equal to the number of such matrices. See Stanley [22] for more details. ♣

The equalities  $M_{\lambda,\mu} = M_{\mu,\lambda}$  and  $N_{\lambda,\mu} = N_{\mu,\lambda}$  are a direct result from the definitions above, which tells us that the matrices of change of basis between  $\{e_\lambda\}$  and  $\{m_\lambda\}$ , and between  $\{h_\lambda\}$  and  $\{m_\lambda\}$  are symmetric.

**Corollary 3.3.**

- $\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda, \mu} N_{\lambda\mu} m_\lambda(\mathbf{x}) m_\mu(\mathbf{y}) = \sum_{\lambda \in \mathcal{P}} m_\lambda(\mathbf{x}) h_\lambda(\mathbf{y}).$
- $\prod_{i,j} (1 + x_i y_j) = \sum_{\lambda, \mu \in \mathcal{P}} M_{\lambda\mu} m_\lambda(\mathbf{x}) m_\mu(\mathbf{y}) = \sum_{\lambda \in \mathcal{P}} m_\lambda(\mathbf{x}) e_\lambda(\mathbf{y}).$

Other important relations that we are going to be using are those between complete, elementary and power sum symmetric functions.

**Proposition 3.2** (see Stanley [22]).

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda} z_\lambda^{-1} p_\lambda(\mathbf{x}) p_\lambda(\mathbf{y}). \quad (3.1)$$

$$\prod_{i,j} (1 + x_i y_j) = \sum_{\lambda} z_\lambda^{-1} \varepsilon_\lambda p_\lambda(\mathbf{x}) p_\lambda(\mathbf{y}). \quad (3.2)$$

where  $z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \cdots r^{m_r} m_r!$ .

*Proof.* (Sketch)

$$\begin{aligned} \log \prod_{i,j} (1 - x_i y_j)^{-1} &= \sum_{i,j} \log((1 - x_i y_j)^{-1}) \\ &= \sum_{n \geq 0} \frac{1}{n} \left( \sum_i x_i^n \right) \left( \sum_j y_j^n \right) \\ &= \sum_{n \geq 0} \frac{1}{n} p_n(\mathbf{x}) p_n(\mathbf{y}). \end{aligned}$$

This result can be used to prove Equation 3.1 (see Stanley [22]). In a similar way we can prove (3.2). ♣

**Corollary 3.4.**

$$h_n = \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda. \quad (3.3)$$

$$e_n = \sum_{\lambda \vdash n} \varepsilon_\lambda z_\lambda^{-1} p_\lambda. \quad (3.4)$$

*Proof.* To prove (3.3), use the first part of Corollary 3.3 and set  $y_1 = t, y_2 = y_3 = \dots = 0$ . Then use formula (3.1). Similarly for (3.4) but using the second part of Corollary 3.3 and formula (3.2) instead. ♣

**Example 3.4.** For example:

$$h_4 = \sum_{\mu \vdash 4} \frac{p_\mu}{z_\mu} = \frac{p_1^4}{24} + \frac{p_1^2 p_2}{4} + \frac{p_1 p_3}{3} + \frac{p_2^2}{8} + \frac{p_4}{4}.$$

Define the following two symmetric series in  $\Lambda(t)$  :

$$h(t) := \sum_{n \geq 0} h_n t^n \quad \text{and} \quad e(t) := \sum_{n \geq 0} e_n t^n.$$

Using Corollary 3.4, the notation  $\lambda = 1^{m_1} 2^{m_2} \dots r^{m_r}$  and the definition of  $z_\lambda$ , these have simple expressions in the power sum basis:

$$h(t) = \sum_{n \geq 0} h_n t^n = \exp \left( \sum_{k \geq 1} \frac{p_k t^k}{k} \right), \quad (3.5)$$

$$e(t) = \sum_{n \geq 0} e_n t^n = \exp \left( \sum_{k \geq 1} (-1)^{k-1} \frac{p_k t^k}{k} \right). \quad (3.6)$$

Particularly, denote  $h = h(1)$  and  $e = e(1)$ .

Some connection coefficients have been conveniently omitted from this section. One of the most complete references on this subject is [22].

### 3.4 Schur functions

Schur functions are considered among the most important families of symmetric functions, mainly due to their connection with representation theory and other algebraic topics, which have led to several generalizations. Here we show one of their most relevant generalizations (skew Schur functions) and a more general algebraic definition which expresses them in terms of the complete homogeneous symmetric functions. Recall from the beginning of this section the combinatorial definition of the Schur symmetric function indexed by a given partition  $\lambda$ . With this definition, it is natural to define Schur functions indexed by skew shapes  $\lambda/\mu$ . These symmetric functions are known as *skew Schur functions*. Formally:

$$s_{\lambda/\mu}(\mathbf{x}) := \sum_{\mathbf{T}} \mathbf{x}^{\mathbf{T}}. \quad (3.7)$$

summed over all SSYTs  $\mathbf{T}$  of shape  $\lambda/\mu$ . Notice that this is a more general form of Schur function, since  $s_{\lambda/(0)} = s_{\lambda}$ .

From the combinatorial definition it is not obvious that  $s_{\lambda}$  is a symmetric function. However, a key theorem due to Jacobi expresses the Schur functions in terms of the complete homogeneous symmetric function basis. For any partition  $\lambda = (\lambda_1, \dots, \lambda_n)$  and  $\mu = (\mu_1, \dots, \mu_n) \subseteq \lambda$ , we have:

$$s_{\lambda/\mu} = \det (h_{\lambda_i - \mu_j - i + j})_{i,j=1}^n.$$

where  $h_0 = 1$  and  $h_k = 0$  for  $k < 0$ . This is known as the *Jacobi-Trudi identity* and it can be proven by either a combinatorial or an algebraic argument [22]. Notice that by using this formula the definition of Schur functions can be extended to weak compositions (that is, to the case where the sequence  $\lambda_1, \lambda_2, \dots$  is not necessarily decreasing).

**Example 3.5.** Consider  $\lambda = (3, 3, 1)$  and  $\mu = (0)$ . We have

$$s_{\lambda} = s_{(3,3,1)/(0)} = \begin{vmatrix} h_3 & h_4 & h_5 \\ h_2 & h_3 & h_4 \\ 0 & 1 & h_1 \end{vmatrix} = h_5 h_2 + h_3^2 h_1 - h_3 h_4 - h_2 h_4 h_1.$$



We remark that expanding the complete symmetric functions in the monomial symmetric functions basis will give

$$\begin{aligned} s_{(3,3,1)} &= m_{(3,3,1)} + m_{(3,2,2)} + 3m_{(2,2,2,1)} + 2m_{(3,2,1,1)} + 6m_{(2,2,1,1,1)} + 3m_{(3,1,1,1,1)} \\ &\quad + 11m_{(2,1,1,1,1,1)} + 21m_{(1,1,1,1,1,1,1)}. \end{aligned}$$

By Equation (3.7), there are 6 SSYT of shape  $(3, 3, 1)$  and type  $(2, 2, 1, 1, 1)$ . Thus, the coefficients of the sum above are the number of SSYT of shape  $(3, 3, 1)$  and type equal to the partition indexing the corresponding monomial symmetric function.

## 3.5 Operations on symmetric functions

As pointed out before, a better understanding of the scalar and inner product of symmetric functions is one of the key aims of algebraic combinatorics, and in general terms the motivation of our work. These two products are defined in the next subsections, along with the notion of plethysm.

### 3.5.1 Plethysm

*Plethysm* is a composition of symmetric functions, which corresponds to a sort of combinatorial composition. Let  $f(\mathbf{x})$  be a symmetric function. Define:

$$p_n[f](\mathbf{x}) := f(x_1^n, x_2^n, \dots).$$

This definition can be extended by linearity and multiplicativity to any symmetric function, since  $\{p_\lambda : \lambda \in \mathcal{P}\}$  forms a basis for  $\Lambda$ . Indeed, given any two symmetric functions  $f, g \in \Lambda$  such that  $f = \sum_\lambda a_\lambda p_\lambda$ , we have:

$$f[g] = \sum_\lambda a_\lambda p_\lambda[g] = \sum_\lambda a_\lambda \prod_{i=1}^{l(\lambda)} g(x_1^{\lambda_i}, x_2^{\lambda_i}, \dots).$$

The following properties are satisfied by plethysm:

- $p_n[f + g] = p_n[f] + p_n[g]$ .
- $p_n[fg] = p_n[f]p_n[g]$ .
- $p_n[c] = c$ , for any constant  $c$ .

- $p_n[p_m] = p_{nm}$ . Moreover,  $p_\lambda[p_m] = p_{m\lambda}$ , for any integer partition  $\lambda$  and any integer  $m \geq 1$ , where  $m\lambda$  denotes the integer partition of  $m|\lambda|$  obtained by multiplying each part of  $\lambda$  by  $m$ .

The following are some useful identities involving plethysm:

**Proposition 3.3** (Important identities I, Macdonald [14]).

$$h[h_2] = \prod_{i \leq j} (1 - x_i x_j)^{-1}, \quad h[e_2] = \prod_{i < j} (1 - x_i x_j)^{-1},$$

$$e[h_2] = \prod_{i \leq j} (1 + x_i x_j) \quad \text{and} \quad e[e_2] = \prod_{i < j} (1 + x_i x_j).$$

where  $h = \sum_{n \geq 0} h_n$  and  $e = \sum_{n \geq 0} e_n$ .

*Proof.* We show that

$$h[e_2] = \sum_{n \geq 0} h_n[e_2] = \prod_{i < j} (1 - x_i x_j)^{-1}.$$

By using Corollary 3.4 we have:

$$\begin{aligned} \sum_{n \geq 0} h_n[e_2] &= \sum_{n \geq 0} \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda[e_2] \\ &= \sum_{n \geq 0} \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda[e_2] p_\lambda[1] \\ &= \sum_{\lambda} z_\lambda^{-1} p_\lambda[e_2] p_\lambda[1] \\ &= \sum_{\lambda} z_\lambda^{-1} p_\lambda(x_1 x_2, x_1 x_3, x_1 x_4, \dots) p_\lambda(1, 0, 0, 0, \dots) \\ &= \prod_{i < j} (1 - x_i x_j)^{-1} \text{ by equation (3.1).} \end{aligned}$$

Similarly, we can prove the rest of the identities. ♣

**Corollary 3.5** (Important identities II, Macdonald [14]).

$$\sum_{\lambda} s_\lambda = h[e_1 + e_2].$$

*Proof.* It is known that

$$\sum_{\lambda} s_\lambda = \prod_i (1 - x_i)^{-1} \prod_{i < j} (1 - x_i x_j)^{-1} \text{ (see Macdonald [14]).}$$

Then it suffices to show that

$$h[e_1 + e_2] = \prod_i (1 - x_i)^{-1} \prod_{i < j} (1 - x_i x_j)^{-1}.$$

Indeed:

$$h[e_1 + e_2] = \sum_{n \geq 0} h_n[e_1 + e_2] = \sum_{n \geq 0} h_n(x_1, x_2, \dots, x_1 x_2, x_1 x_3, \dots).$$

Since any monomial in  $\{x_1, \dots, x_1 x_2, \dots\}$  is a monomial in  $\{x_1, x_2, \dots\}$ , times a monomial in  $\{x_1 x_2, x_1 x_3, \dots\}$ , then:

$$\sum_{n \geq 0} h_n(x_1, x_2, \dots, x_1 x_2, x_1 x_3, \dots) = \left( \sum_{n \geq 0} h_n[e_1] \right) \left( \sum_{n \geq 0} h_n[e_2] \right).$$

By the previous proposition  $\sum_{n \geq 0} h_n[e_2] = \prod_{i < j} (1 - x_i x_j)^{-1}$ . On the other hand:

$$\begin{aligned} \sum_{n \geq 0} h_n[e_1] &= \sum_{n \geq 0} h_n \\ &= \sum_{n \geq 0} \sum_{\lambda \vdash n} z_\lambda^{-1} p_\lambda \\ &= \sum_{\lambda} z_\lambda^{-1} p_\lambda(x_1, x_2, \dots) p_\lambda(1, 0, 0, \dots) \\ &= \prod_i \frac{1}{1 - x_i}. \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{n \geq 0} h_n[e_1 + e_2] &= \left( \sum_{n \geq 0} h_n[e_1] \right) \left( \sum_{n \geq 0} h_n[e_2] \right) \\ &= \prod_i (1 - x_i)^{-1} \prod_{i < j} (1 - x_i x_j)^{-1}, \end{aligned}$$

and the result follows. ♣

### 3.5.2 Scalar product of Symmetric functions

The *scalar product* on  $\Lambda$  is a symmetric bilinear operation  $\langle, \rangle : \Lambda \times \Lambda \rightarrow \mathbb{Q}$  such that  $\{m_\lambda\}$  and  $\{h_\mu\}$  are *adjoint* bases, i.e:

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda\mu} = 1 \text{ if } \lambda = \mu \text{ and } 0 \text{ otherwise.}$$

Also it can be shown that:

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda, \text{ and } \langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}.$$

where  $z_\mu = 1^{m_1} m_1! 2^{m_2} m_2! \cdots r^{m_r} m_r!$  for  $\mu = 1^{m_1} 2^{m_2} \cdots r^{m_r}$ . Notice that  $\{p_\lambda\}$  and  $\{p_\lambda/z_\lambda\}$  are also adjoint bases.

Recall that any symmetric function can be written as a linear combination of the monomial basis, and that  $\langle f, h_\lambda \rangle$  is the coefficient of  $m_\lambda$  in the monomial basis expansion of  $f$ ; thus the scalar product can be used to get coefficients of a particular monomial in a given symmetric function. Also using the orthogonality of the power sums, one can produce explicit formulas for the coefficients of some monomials. These facts provide some methods of coefficient extraction. We will state in Section 5.4 some of the results that Gessel [7] has obtained using such methods applied to interesting enumerative problems.

### 3.5.3 Application to Schur functions

From the definition of Schur functions given by Equation (3.7), we can write the skew Schur functions in terms of the monomial basis as follows:

$$s_{\lambda/\mu} = \sum_{\nu} K_{\lambda/\mu}^{\nu} m_{\nu}$$

where  $K_{\lambda/\mu}^{\nu}$  denotes the number of SSYT of shape  $\lambda/\mu$  and type  $\nu$ . These coefficients are known as the *skew Kostka numbers*. It is possible to write these coefficients as a scalar product using the orthogonality of the monomial and complete homogeneous symmetric functions as follows:

$$K_{\lambda/\mu}^{\nu} = \langle s_{\lambda/\mu}, h_{\nu} \rangle.$$

For example  $K_{\lambda}^{1^n} = \langle s_{\lambda}, h_1^n \rangle$ , is the number of SYT of shape  $\lambda$ . The generating function of the class of all SYT is then,

$$Y(t) = \sum_n \left( \sum_{\lambda \vdash n} K_{\lambda}^{1^n} \right) t^n = \sum_n \left\langle \sum_{\lambda \vdash n} s_{\lambda}, h_1^n \right\rangle t^n = \left\langle \sum_{\lambda} s_{\lambda}, \sum_n h_1^n t^n \right\rangle. \quad (3.8)$$

This generating series was studied by Bender and Knuth [3] and Gordon [8] in the case that  $\lambda$  is of bounded height. Their work gave a closed form in terms of Bessel functions. Gessel connected these results with his definition of D-finite symmetric functions and obtained equivalent results for bounded height partitions. But we will see more on this generating function in Chapter 5 where we study the D-finiteness of different SYT generating functions.

### 3.5.4 Kronecker product of symmetric functions

The remaining relevant operation of symmetric functions that we work with is called the *Kronecker product*, also known as the *inner product*, which was first introduced in 1927 by Redfield [19] and is defined by:

$$p_\lambda * p_\mu = \delta_{\lambda\mu} z_\lambda p_\lambda.$$

Notice that if we set  $p_{\lambda_i} = 1$  for all  $i$  after applying the Kronecker product we get the scalar product. This idea is going to be useful later. Also, this product can be extended by linearity. In particular, for Schur functions, we have that

$$s_\lambda * s_\mu = \sum_{\nu} C_{\lambda\mu}^{\nu} s_{\nu},$$

for some positive integer coefficients  $C_{\lambda\mu}^{\nu}$  which have been studied by Murnaghan for particular shapes. There are currently no explicit formulas for these coefficients. We are going to study this with more detail in Chapter 6.

**Example 3.6.** Consider the following Kronecker products of Schur functions,

- $s_{(2)} * s_{(1,1)} = s_{(1,1)}$ .
- $s_{(4,2,1,1)} * s_{(4,4)} = s_{(6,1,1)} + s_{(5,3)} + 2s_{(5,2,1)} + s_{(5,1,1,1)} + 2s_{(4,3,1)} + s_{(4,2,2)} + 4s_{(4,2,1,1)} + s_{(4,1,1,1,1)} + 2s_{(3,3,2)} + s_{(3,3,1,1)} + 2s_{(3,2,2,1)} + 2s_{(3,2,1,1,1)} + s_{(3,1,1,1,1,1)} + s_{(2,2,2,1,1)}$ .

**Remark 3.1.** It is worth pointing out that  $\Lambda$  with the Kronecker product forms a commutative ring with identity  $h = \sum_n h_n$ . In fact:

$$\begin{aligned} h * p_\mu &= \left( \sum_{\lambda} \frac{p_\lambda}{z_\lambda} \right) * p_\mu \\ &= \sum_{\lambda} \frac{p_\lambda * p_\mu}{z_\lambda} \\ &= \sum_{\lambda} \frac{\delta_{\lambda\mu} z_\lambda p_\lambda}{z_\lambda}. \end{aligned}$$

Therefore,

$$h * p_\mu = p_\mu. \tag{3.9}$$

## Chapter 4

# D-finite functions

We already defined rational and algebraic generating functions in Section 2.1. Here we focus our attention on the more general D-finite (differentiably finite) generating functions in one variable, which were introduced by Stanley [21] in 1980. Two years later this concept was generalized to the multivariate case by Zeilberger [25]. And after ten years, Gessel [7] developed a D-finiteness theory for symmetric series. In this chapter we recall all these definitions and some important results concerning the closure properties of D-finiteness with respect to scalar and inner products.

### 4.1 Definitions

Consider any field  $\mathbb{Q}$  with characteristic 0 and  $\mathbb{Q}[[x]]$  the ring of formal power series in  $x$ . We say that a power series  $F \in \mathbb{Q}[[x]]$  is *D-finite* (differentiably finite) in a variable  $x$  if  $F$  and all of its derivatives  $F^{(n)} = \frac{d^n F}{dx^n}$  ( $n \geq 1$ ) span a finite-dimensional vector subspace of  $\mathbb{Q}[[x]]$  over the field  $\mathbb{Q}(x)$  of rational functions. It is proven in [22] that this is equivalent to saying that there exist finitely many polynomials  $q_0(x), \dots, q_k(x)$ , with  $q_k \neq 0$ , such that:

$$q_k(x)F^{(k)} + q_{k-1}(x)F^{(k-1)} + \dots + q_0(x)F = 0. \quad (4.1)$$

where  $F^{(k)}$  denotes the  $k$ -th derivative of  $F$ .

As with the rational and algebraic generating functions, it turns out that the coefficients of a D-finite generating function satisfy a recurrence relation. A function  $f : \mathbb{N} \rightarrow \mathbb{Q}$  is *P-recursive* (polynomially recursive) if there exists a finite number of polynomials

$a_0(n), \dots, a_k(n)$ , with  $a_k \neq 0$ , such that for all  $n \in \mathbb{N}$ :

$$a_k(n)f(n+k) + a_{k-1}(n)f(n+k-1) + \dots + a_0(n)f(n) = 0. \quad (4.2)$$

**Proposition 4.1.** *The power series  $F = \sum_n f(n)x^n \in \mathbb{Q}[[x]]$  is D-finite if and only if  $f(n)$  is P-recursive.*

The proof of the only-if implication is obtained by noticing that the coefficient of  $x^{n+k}$  in the left hand side of equation (4.1) is equal to the right hand side of equation (4.2), for some polynomials  $a_i$  that depend on the  $q_i$ . The other implication is proven by using the fact that  $(n+i)_j$  (the  $j$ th descending factorial of  $(n+i)$ ), for  $j \geq 0$  forms a  $\mathbb{Q}$ -basis for the space  $\mathbb{Q}[n]$  and so each  $a_i(n)$  (equation 4.2) will be a linear combination of them (see Stanley[22]).

**Example 4.1.** Consider the formal power series  $F(x) = \exp(x) = \sum_{n \geq 0} \frac{x^n}{n!}$ . We have that  $F(x)$  is D-finite since all its derivatives are equal to  $\exp(x)$ , so they form a 1-dimensional vector space. Equivalently,  $F(x)$  satisfies the differential equation  $F'(x) - F(x) = 0$ . Also,  $f(n) = 1/n!$  is P-recursive, since  $nf(n) - f(n-1) = 0$ .

Using a similar argument we can prove the following proposition:

**Proposition 4.2.** *Let  $p(x)$  be a polynomial in  $\mathbb{Q}[x]$ . Then  $\exp(p(x))$  is D-finite.*

We can extend the definition of D-finite power series to multiple variables  $x_1, \dots, x_n$ . To do this, consider the ring  $\mathbb{Q}[[x_1, x_2, \dots, x_n]]$  of power series in  $x_1, \dots, x_n$ . A function  $F \in \mathbb{Q}[[x_1, x_2, \dots, x_n]]$  is *D-finite in the variables  $x_1, x_2, \dots, x_n$*  if the set of all its partial derivatives  $\frac{\partial^{i_1+i_2+\dots+i_n} F}{\partial x_1^{i_1} \partial x_2^{i_2} \dots \partial x_n^{i_n}}$  spans a finite-dimensional vector space. Moreover, a formal power series  $F$  is *D-finite over an infinite number of variables  $\Omega$*  if for any subset of variables  $S \subset \Omega$  the function  $F$  obtained after setting  $x = 0$  for all  $x \in \Omega - S$  is D-finite. It can be shown that a function  $F \in \mathbb{Q}[[x_1, x_2, \dots, x_n]]$  is D-finite if and only if  $F$  satisfies a system of  $n$  linear differential equations with polynomial coefficients, each equation having partial derivatives with respect to only one of the variables  $x_1, x_2, \dots, x_n$ . That is:

$$q_{i,k} \frac{\partial^k F}{\partial x_i^k} + q_{i,k-1} \frac{\partial^{k-1} F}{\partial x_i^{k-1}} + \dots + q_{i,0} F = 0, \quad i = 1, 2, \dots, n, \quad (4.3)$$

for some polynomials  $q_{i,j}$  in the variables  $x_1, x_2, \dots, x_n$  such that not all of the polynomials  $q_{i,0}, q_{i,1}, \dots, q_{i,k}$  are zero.

The combinatorial application below is related to the hook formula. Given the Young diagram  $\mathbf{T} = \mathbf{T}(\mu)$  corresponding to a partition  $\mu$ , and a cell  $x \in \mathbf{T}$ , denote by  $\text{hook}(x)$  the *hook length* of  $x$  (the number of cells in the same row to the right of  $x$  or in the same column below  $x$ , plus  $x$  itself). Then,  $K_\mu^{1^n}$  the number of standard Young tableaux of shape  $\mu$  is equal to

$$K_\mu^{1^n} = \frac{n!}{\prod_{x \in \mathbf{T}} \text{hook}(x)},$$

where  $K_\mu^{1^n}$  is the Kostka number of shape  $\mu$  and type  $1^n$ . This formula has been proven in many different ways, from both algebraic and combinatorial points of view. We direct the reader to the recent proof [2] by Bandlow.

**Proposition 4.3.** *Denote by  $a_m$  the number of standard Young tableaux of shape  $(m-|\lambda|, \lambda)$ , for a fixed partition  $\lambda$ . Then  $\{a_m\}_{m \geq 0}$  is  $P$ -recursive.*

*Proof.* Using the hook formula we get:

$$\begin{aligned} a_m &= \frac{m!}{\prod_{x \in \mathbf{T}(m-|\lambda|, \lambda)} \text{hook}(x)} \\ &= \frac{m!}{\prod_{i=1}^{m-|\lambda|} (\lambda'_i + m - |\lambda| - i + 1) \prod_{x \in \mathbf{T}(\lambda)} \text{hook}(x)} \\ &= \frac{K_\lambda^{1^{|\lambda|}}}{|\lambda|!} \cdot \frac{m!}{\prod_{i=1}^{m-|\lambda|} (\lambda'_i + m - |\lambda| - i + 1)}. \end{aligned}$$

Set  $C_\lambda = \frac{K_\lambda^{1^{|\lambda|}}}{|\lambda|!}$ . Then,

$$\begin{aligned} a_m &= C_\lambda \frac{m!}{\prod_{i=1}^{l(\lambda')} (\lambda'_i + m - |\lambda| - i + 1) \prod_{i=l(\lambda')+1}^{m-|\lambda|} (m - |\lambda| - i + 1)} \\ &= C_\lambda \frac{m!}{\left( \prod_{i=1}^{\lambda_1} (\lambda'_i + m - |\lambda| - i + 1) \right) (m - |\lambda| - \lambda_1)!} \frac{(|\lambda| + \lambda_1)!}{(|\lambda| + \lambda_1)!}, \text{ using } l(\lambda') = \lambda_1. \\ &= D_\lambda \binom{m}{|\lambda| + \lambda_1} \frac{1}{\prod_{i=1}^{\lambda_1} (\lambda'_i + m - |\lambda| - i + 1)}, \text{ for } D_\lambda = C_\lambda (|\lambda| + \lambda_1)! \end{aligned}$$



We claim that any rational sequence in one variable is P-recursive. To see this, suppose that  $w(n) = \frac{u(n)}{v(n)}$ , where  $u$  and  $v$  are polynomials. Then we have  $u(n-1)v(n)w(n) = u(n)v(n-1)w(n-1)$  which proves that  $w(n)$  is P-recursive. Hence  $a_m$ , which is a rational function on  $m$ , is P-recursive and therefore its generating function is D-finite. The linear recursion for  $a_m$  is given by:

$$D_\lambda \binom{m-1}{|\lambda| + \lambda_1} \prod_{i=1}^{\lambda_1} (\lambda'_i + m - |\lambda| - i + 1) a_m = D_\lambda \binom{m}{|\lambda| + \lambda_1} \prod_{i=1}^{\lambda_1} (\lambda'_i + m - |\lambda| - i) a_{m-1}.$$

which can be simplified by removing the constant factor  $D_\lambda$  on both sides and by expanding the binomial coefficients:

$$(m - |\lambda| - \lambda_1) \prod_{i=1}^{\lambda_1} (\lambda'_i + m - |\lambda| - i + 1) a_m = m \prod_{i=1}^{\lambda_1} (\lambda'_i + m - |\lambda| - i) a_{m-1}.$$

Notice that the generating function for  $a_m$  is given by

$$\begin{aligned} \mathcal{K}_\lambda(t) &= \sum_m a_m t^m = \sum_{n,m} \langle s_{n,\lambda}, h_1^m \rangle t^m \\ &= \left\langle \sum_n s_{n,\lambda}, \sum_m h_1^m t^m \right\rangle. \end{aligned}$$

Because the coefficients are P-recursive, this generating function is D-finite in  $t$ .



We find a more general consequence of Proposition 4.3 in Chapter 5.

**Example 4.2.** For the case  $\lambda = (2, 1)$ , the previous recurrence becomes,

$$(m-5)(m-1)(m-3)a_m = m(m-2)(m-4)a_{m-1}.$$

Using Maple we can find the differential equation satisfied by the generating function  $f(t)$ , which in this case is given by,

$$(-3t - 15)\mathcal{K}_{(2,1)}(t) + (3t^2 + 15t) \frac{d\mathcal{K}_{(2,1)}(t)}{dt} - 6t^2 \frac{d^2\mathcal{K}_{(2,1)}(t)}{dt^2} + (t^3 - t^4) \frac{d^3\mathcal{K}_{(2,1)}(t)}{dt^3} = 0.$$

## 4.2 Closure Properties

As we said before, D-finiteness gives information about the simplicity of the coefficients of a power series. If a power series in one variable is D-finite, then its coefficients satisfy a linear recursion with polynomial coefficients, which means that there exists a closed formula or a polynomial time algorithm to calculate these coefficients. This makes D-finiteness very important in enumerative combinatorics. Operations between power series often have a combinatorial meaning, whether it is just coefficient extraction or more complex operations between combinatorial classes, which makes the study of closure properties of D-finiteness a natural and very relevant addition to the theory of D-finite power series.

We denote by  $\mathcal{D}$  the set of all D-finite power series in  $x$ . The following theorem summarizes some of the operations under which D-finiteness is preserved.

**Theorem 4.1** (Closure properties, univariate case. Stanley[22]).

1. If  $F(x), G(x) \in \mathcal{D}$  then  $\alpha F(x) + \beta G(x) \in \mathcal{D}$  and  $F(x)G(x) \in \mathcal{D}$ , for  $\alpha, \beta \in \mathbb{Q}$ ;
2. If  $F(x)$  is algebraic then  $F(x) \in \mathcal{D}$ ;
3. If  $F(x) \in \mathcal{D}$  and  $G(x)$  is an algebraic power series with  $G(0) = 0$ , then  $F(G(x)) \in \mathcal{D}$ .

Proving that a power series is D-finite can be done by either finding the explicit differential equations or by showing that the space where the derivatives of the function lie is finite-dimensional. For the first part, the proof is done by showing that the dimensions of the spaces containing the derivatives of  $\alpha F(x) + \beta G(x)$  and those of  $F(x)G(x)$  are both finite. In the same manner we can prove parts 2 and 3 (see Stanley [22]).

**Example 4.3.** Take  $F(x) = \exp(x)$  and  $G(x) = \frac{1}{\sqrt{1-x}}$ . Using part 1 of Theorem 4.1, we have that  $L(x) = \frac{\exp(x)}{\sqrt{1-x}}$  is D-finite.

From the second part of Theorem 4.1 we have that the coefficients of an algebraic power series are P-recursive as was mentioned in Section 1.

**Theorem 4.2** (Closure properties, multivariate case. Zeilberger[25]).

1. If  $F$  is D-finite with respect to  $x_1, x_2, \dots, x_n$ , then  $F$  is D-finite with respect to any subset of  $x_1, x_2, \dots, x_n$ . The same holds when  $F$  is D-finite with respect to an infinite number of variables.

2. If  $F(x_1, x_2, \dots, x_n)$  is D-finite in  $x_1, x_2, \dots, x_n$  and for each  $k$ ,  $u_k$  is a polynomial in the variables  $y_1, y_2, \dots, y_m$ , then  $F(u_1, u_2, \dots, u_n)$  (as long as it is well-defined as a formal power series) is D-finite in  $y_1, y_2, \dots, y_m$ . In other words we may replace the variables  $x_1, \dots, x_n$  for polynomials in another finite set of variables preserving D-finiteness. This holds also when  $F$  is D-finite with respect to an infinite number of variables, but the replacement can only be made on a finite subset of these variables.

These statements can be proven using the same type of techniques used for the univariate closure properties (Theorem 4.1).

The *Hadamard product* ( $\odot$ ) of two power series  $F(\mathbf{x}) = \sum_{\mathbf{i}} a_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}$  and  $G(\mathbf{x}) = \sum_{\mathbf{j}} b_{\mathbf{j}} \mathbf{x}^{\mathbf{j}}$  (where  $\mathbf{i} = (i_1, \dots, i_n)$  and  $\mathbf{x}^{\mathbf{i}} = x_1^{i_1} x_2^{i_2} \dots x_n^{i_n}$ ) with respect to  $\mathbf{x}$  is defined as follows:

$$F(\mathbf{x}) \odot G(\mathbf{x}) := \sum_{\mathbf{i}} a_{\mathbf{i}} b_{\mathbf{i}} \mathbf{x}^{\mathbf{i}}.$$

What is the effect of the Hadamard product on D-finite power series? This question was answered by Lipshitz in 1988:

**Lemma 4.1** (Lipshitz [12]). *Suppose that  $F$  and  $G$  are D-finite in  $x_1, x_2, \dots, x_{m+n}$ . Then the Hadamard product  $F \odot G$  with respect to the variables  $x_1, x_2, \dots, x_n$  is D-finite in  $x_1, x_2, \dots, x_{m+n}$ .*

In this section we have presented a reasonable collection of tools that allow us to show that a given power series or generating function is D-finite. However, we do not have many general techniques to prove that a formal power series is not D-finite. Usually a good argument is by contradiction, as illustrated in Example 4.4.

**Example 4.4.** Consider the function  $F(x) = \exp(\exp(x))$ . We claim that  $F(x)$  is not D-finite. Suppose, in order to obtain a contradiction, that  $F(x)$  is D-finite. Thus the vector space  $V$  spanned by the derivatives of all orders of  $F(x)$  is of finite dimension  $d$ . We claim that  $\exp(nx) \exp(\exp(x)) \in V$  for all  $n \in \{0, 1, 2, \dots\}$ . Clearly  $\exp(0x) \exp(\exp(x)) = F(x) \in V$ . Inductively if  $\exp(nx) \exp(\exp(x)) \in V$ , then its derivative  $n \exp(nx) \exp(\exp(x)) + \exp((n+1)x) \exp(\exp(x))$  is also in  $V$ , and so  $\exp((n+1)x) \exp(\exp(x)) \in V$ . Hence there is a nontrivial linear combination of  $\{\exp(nx) \exp(\exp(x))\}_{0 \leq n \leq d}$  which is equal to 0. That is:

$$\left( \sum_{n=0}^d a_n \exp(nx) \right) \exp(\exp(x)) = 0,$$

for some scalars  $a_n$ . Since  $\exp(\exp(x)) \neq 0$ , we have that  $\exp(x)$  satisfies a nontrivial polynomial of degree  $d$ , which implies that  $\exp(x)$  is algebraic, contradicting an example mentioned in Section 2.2.

There exists another argument based on singularity analysis to prove that a function is not D-finite. As an example of this, consider:

**Theorem 4.3** (Folklore). *The number of partitions of  $n$ , denoted by  $\widehat{p}(n)$ , is not P-recursive.*

In order to prove this, it suffices to show that the ordinary generating series

$$\sum_{n \geq 0} \widehat{p}(n)t^n = \prod_k \frac{1}{1-t^k}$$

is not D-finite. For this, notice that such generating function has an infinite number of singularities, which is not possible for a D-finite power series. We do not cover this type of analysis in our work, the reader is directed to Flajolet and Sedgewick [6] for further details.

### 4.3 Symmetric Series

The last extension of this definition applies to symmetric functions. Consider the power sum symmetric functions as formal variable and the ring  $\mathbb{Q}[[p_1, p_2, \dots, p_n, \dots]]$  of symmetric power series. A symmetric series  $f \in \mathbb{Q}[[p_1, p_2, \dots, p_n, \dots]]$  is D-finite if it is D-finite with respect to any finite subset of the  $p_i$ 's, after setting  $p_j = 0$  for every  $p_j$  that is not in the subset. Equivalently, after setting  $p_i = 0$  for  $i \geq n+1$ , the set of all its partial derivatives with respect to the  $p_i$ 's spans a finite-dimensional vector subspace of  $\mathbb{Q}[[p_1, p_2, \dots, p_n]]$ . Moreover, if we consider the ring  $\mathbb{Q}[[t, p_1, p_2, \dots, p_n, \dots]]$ , then a symmetric series  $f \in \mathbb{Q}[[t, p_1, p_2, \dots, p_n, \dots]]$  is D-finite if it is D-finite with respect to  $t$  and the  $p_i$ 's.

**Example 4.5.** Recall that  $h = \sum_{n \geq 0} h_n = \exp\left(\sum_{k \geq 1} \frac{p_k}{k}\right)$  (Equation 3.5). If we set  $p_k = 0$  for  $k > n_0$  for some  $n_0 \in \mathbb{N}$ , we get

$$h|_{n_0} = \exp\left(\sum_{k=1}^{n_0} \frac{p_k}{k}\right)$$

which is clearly D-finite with respect the  $p_i$ 's. Notice that  $F = \exp(h)$  is D-finite with respect the  $h_i$ 's, but we cannot say the same with respect to the  $p_i$ 's, because

$$F = \exp\left(\exp\left(\sum_{k \geq 1} \frac{p_k}{k}\right)\right),$$

is not D-finite in the  $p_i$ 's (set  $p_k = 0$  for  $k \geq 2$  to obtain  $\exp(\exp(p_1))$  which is not D-finite by Example 4.4).

**Proposition 4.4.** *Let  $g \in \mathbb{Q}[t, p_1, p_2, \dots, p_n]$  then  $\exp(g)$  is D-finite with respect to  $t$  and the  $p_i$ 's.*

The proof is straightforward since the derivative of a polynomial is a polynomial.

**Example 4.6.** For instance, the symmetric series,

$$w(t) = \exp\left(\sum_{k=1}^m \frac{p_k t^k}{k}\right)$$

is D-finite with respect to  $t$  and the  $p_i$ 's. The system of differential equations satisfied by  $w(t)$  is given by:

$$\begin{cases} \frac{\partial w(t)}{\partial p_k} - w(t) \frac{t^k}{k} = 0 & \text{for } 1 \leq k \leq m, \\ \frac{\partial w(t)}{\partial t} - w(t) \sum_{i=1}^m t^{i-1} p_i = 0. \end{cases}$$

## 4.4 Plethysm

Next we present some examples and results that reflect the relations between the notion of plethysm and the theory of D-finiteness of symmetric functions.

**Theorem 4.4** (Gessel[7]). *If  $g$  is a symmetric polynomial then  $h[g]$  and  $e[g]$  are D-finite.*

*Proof.* Let  $g$  be a symmetric polynomial of degree  $d$ . Hence, we can write  $g$  in terms of the power sum basis  $g = \sum_{|\lambda| \leq d} a_\lambda p_\lambda$ . We obtain:

$$\begin{aligned} h[g] = \sum_{n \geq 0} h_n[g] &= \exp\left(\sum_{k \geq 0} \frac{p_k[g]}{k}\right), \text{ since } h = \exp\left(\sum_k \frac{p_k}{k}\right). \\ &= \exp\left(\sum_{k \geq 0} \frac{g[p_k]}{k}\right) \\ &= \exp\left(\sum_{k \geq 0} \frac{g(x_1^k, x_2^k, \dots)}{k}\right). \end{aligned}$$

Let us write  $g(x_1^k, x_2^k, \dots) = \sum_{|\lambda| < d} a_\lambda p_\lambda(x_1^k, x_2^k, \dots)$ . Then:

$$h[g] = \exp \left( \sum_{k \geq 0} \sum_{|\lambda| < d} \frac{a_\lambda p_{k\lambda}}{k} \right).$$

Where  $k\lambda = (k\lambda_1, k\lambda_2, \dots, k\lambda_l)$  for any given partition  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ . Now set  $p_i = 0$  for  $i > m$ :

$$h[g]|_{p_i=0, i>m} = \exp \left( \sum_{k=0}^m \sum_{|\lambda| < d} \frac{a_\lambda p_{k\lambda}}{k} \right).$$

Since  $p_{k\lambda} = p_{k\lambda_1} \cdots p_{k\lambda_l}$  and hence  $p_{k\lambda} = 0$  for  $k > m$ . Thus the symmetric series  $\sum_{n \geq 0} h_n[g]$  is D-finite in  $\{p_1, p_2, \dots, p_m\}$  by Proposition 4.4 and therefore it is D-finite in  $\{p_1, p_2, \dots\}$ . ♣

**Example 4.7.** Some examples of D-finite symmetric functions using plethysm are as follows:

- Consider the symmetric polynomials  $h_2 = \frac{p_1^2 + p_2}{2}$ . and  $e_2 = \frac{p_1^2 - p_2}{2}$ . Recall the identities:

$$h[h_2] = \prod_{i \leq j} (1 - x_i x_j)^{-1}, \quad h[e_2] = \prod_{i < j} (1 - x_i x_j)^{-1},$$

$$e[h_2] = \prod_{i \leq j} (1 + x_i x_j), \quad e[e_2] = \prod_{i < j} (1 + x_i x_j).$$

These symmetric series are all D-finite in the  $p_i$ 's.

- Also,  $\sum_\lambda s_\lambda = h[e_1 + e_2]$  is D-finite. More generally, for a fixed partition  $\mu$ ,  $\sum_\lambda s_{\lambda/\mu} = h[e_1 + e_2] \sum_\lambda s_{\mu/\lambda}$  is D-finite.

These identities were shown in Subsection 3.5.1. The coefficient of each monomial in the expansion of these symmetric series has a nice combinatorial interpretation. We will state this in Chapter 5, where we relate these coefficients to a graph enumeration problem that was studied by Goulden and Jackson [11].

## 4.5 Schur Functions

In this subsection, we present some examples of D-finite symmetric series that were not considered by Gessel. They play an important role in Chapters 5 and 6.

**Lemma 4.2.** *For a fixed partition  $\lambda$ , the symmetric series  $\sum_m s_{m-|\lambda|,\lambda} t^m = \sum_n s_{n,\lambda} t^{n+|\lambda|}$  is D-finite in  $\{t, p_1, p_2, \dots\}$ .*

*Proof.* Let  $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$  be a partition of length  $r$ . Recall that  $s_\mu = \det (h_{\mu_i-i+j})_{i,j=1}^n$ . Thus:

$$\begin{aligned} s_{n,\lambda} &= \begin{vmatrix} h_n & h_{n+1} & \dots & h_{n+r} \\ h_{\lambda_1-1} & h_{\lambda_1} & \dots & h_{\lambda_1+r-1} \\ \vdots & \dots & \ddots & \vdots \\ h_{\lambda_r-r} & h_{\lambda_r-r+1} & \dots & h_{\lambda_r} \end{vmatrix} \\ &= h_n s_\lambda - h_{n+1} s_{\lambda/1} + h_{n+2} s_{\lambda/1^2} - \dots + (-1)^r h_{n+r} s_{\lambda/(1)^r}. \end{aligned}$$

As it was observed by Thibon[24],

$$\begin{aligned} \sum_n s_{n,\lambda} t^{n+|\lambda|} &= t^{|\lambda|} \sum_n (h_n s_\lambda - h_{n+1} s_{\lambda/1} + h_{n+2} s_{\lambda/1^2} - \dots + (-1)^r h_{n+r} s_{\lambda/(1)^r}) t^n \\ &= t^r \sum_{k,m} (-1)^k h_m s_{\lambda/1^k} t^m \\ &= t^r \left( \sum_m h_m t^m \right) \left( \sum_k (-1)^k s_{\lambda/1^k} t^\lambda \right) \\ &= t^r h(t) \sum_{k=0}^r (-1)^k s_{\lambda/1^k}. \end{aligned}$$

Now we recall that  $h(t)$  is D-finite with respect to  $t$  and the  $p_i$ 's. Also,  $\sum_{k=0}^r (-1)^k s_{\lambda/1^k}$  is D-finite in the  $p_i$ 's, because it is a finite polynomial in the Schur functions, which implies that it is also finite in terms of the  $p_i$ 's. Therefore  $\sum_n s_{n,\lambda} t^{n+|\lambda|}$  is D-finite in  $\{t, p_1, p_2, \dots\}$ . ♣

**Example 4.8.** Set  $\lambda = 1$  in the previous lemma:

$$\begin{aligned} \sum_{n \geq 0} s_{n-1,1} t^n &= \sum_{n \geq 0} (h_{n-1} h_1 - h_n) t^n \\ &= (p_1 t - 1) \sum_{n \geq 0} h_n t^n \\ &= (p_1 t - 1) \exp \left( \sum_{k \geq 1} t^k \frac{p_k}{k} \right) \\ &= (p_1 t - 1) h(t), \end{aligned}$$

which is D-finite, because it is the product of two D-finite symmetric functions.

**Lemma 4.3.** For a fixed nonnegative integer  $k$ , the symmetric series  $\sum_n s_{n^k} t^{kn}$  is D-finite in  $\{t, p_1, p_2, \dots\}$ .

*Proof.* Let  $\mu$  be a partition of length  $n$ . Recall that  $s_\mu = \det(h_{\mu_i - i + j})_{i,j=1}^n$ . Let  $\mathcal{S}_n$  be the permutation group of  $n$  elements. Denote by  $\text{sgn}(\sigma)$  the sign of a permutation  $\sigma \in \mathcal{S}_n$ .

$$\begin{aligned} \sum_n s_{n^k} t^{kn} &= \sum_n \det(h_{n-i+j})_{i,j=1}^k t^{kn} \\ &= \sum_n \left( \sum_{\sigma \in \mathcal{S}_k} \text{sgn}(\sigma) \prod_{i=1}^k h_{n-i+\sigma(i)} \right) t^{kn} \\ &= \sum_{\sigma \in \mathcal{S}_k} \text{sgn}(\sigma) \sum_n \left( \prod_{i=1}^k h_{n-i+\sigma(i)} \right) t^{kn}. \end{aligned}$$

Using the Hadamard product we can write the sum over  $n$  as  $k$  Hadamard products.

$$\sum_n \left( \prod_{i=1}^k h_{n-i+\sigma(i)} \right) t^{kn} = \bigodot_{i=1}^k \left( \sum_n h_{n-i+\sigma(i)} t^{kn} \right).$$

Now,

$$\sum_n h_{n-i+\sigma(i)} t^{kn} = t^{ki-k\sigma(i)} \sum_m h_m t^{km} = t^{ki-k\sigma(i)} h(t^k),$$

D-finite with respect to  $t$  and the  $p_i$ 's, since the  $t \rightarrow t^k$  preserves D-finiteness. The result follows from Theorem 4.1. ♣

**Example 4.9.** Set  $k = 2$  in the previous Lemma:

$$\begin{aligned} \sum_{n \geq 0} s_{(n,n)} t^{2n} &= \sum_{n \geq 0} (h_n^2 - h_{n-1} h_{n+1}) t^{2n} \\ &= \sum_{n \geq 0} h_n^2 t^{2n} - \sum_{n \geq 0} h_{n-1} h_{n+1} t^{2n}. \end{aligned}$$

We have that

$$\sum_{n \geq 0} h_n^2 t^{2n} = \sum_{n \geq 0} h_n t^{2n} \odot \sum_{n \geq 0} h_n t^{2n}$$

is D-finite. On the other hand, since

$$\sum_{n \geq 0} h_{n-1} h_{n+1} t^{2n} = \sum_{n \geq 0} h_{n-1} t^{2n} \odot \sum_{n \geq 0} h_{n+1} t^{2n},$$

is D-finite because  $\sum_{n \geq 1} h_{n-1} t^{2n} = t^2 \sum_{m \geq 0} h_m t^{2m}$  and  $\sum_{n \geq 1} h_{n+1} t^{2n} = \frac{1}{t^2} \sum_{m \geq 1} h_m t^{2m}$  are D-finite, then  $\sum_{n \geq 0} s_{(n,n)} t^{2n}$  is a D-finite symmetric series.



**Theorem 4.5.** *Let  $\lambda$  and  $\mu$  be fixed partitions and  $k$  a fixed nonnegative integer. Then the symmetric series  $\sum_n s_{(n^k, \lambda)/\mu} t^{kn+|\lambda|-|\mu|}$  is D-finite in  $\{t, p_1, p_2, \dots\}$ .*

*Proof.* Set  $\rho_i = n$  if  $1 \leq i \leq k$  and  $\rho_i = \lambda_{i-k}$  for  $k+1 \leq i \leq k+r$ . We obtain

$$\begin{aligned} & \sum_n s_{(n^k, \lambda)/\mu} t^{kn+|\lambda|-|\mu|} = \sum_n \det (h_{\rho_i - \mu_j - i + j})_{i,j=1}^{k+r} t^{kn+|\lambda|-|\mu|} \\ &= \sum_n \left( \sum_{\sigma \in S_{k+r}} \operatorname{sgn}(\sigma) \prod_{i=1}^k h_{n - \mu_{\sigma(i)} - i + \sigma(i)} \prod_{i=k+1}^{k+r} h_{\lambda_{i-k} - \mu_{\sigma(i)} - i + \sigma(i)} \right) t^{kn+|\lambda|-|\mu|} \\ &= \sum_{\sigma \in S_{k+r}} \operatorname{sgn}(\sigma) \sum_n \left( \prod_{i=1}^k h_{n - \mu_{\sigma(i)} - i + \sigma(i)} \prod_{i=k+1}^{k+r} h_{\lambda_{i-k} - \mu_{\sigma(i)} - i + \sigma(i)} \right) t^{kn+|\lambda|-|\mu|}. \end{aligned}$$

Let us denote

$$R_\sigma = \sum_n \left( \prod_{i=1}^k h_{n - \mu_{\sigma(i)} - i + \sigma(i)} \prod_{i=k+1}^{k+r} h_{\lambda_{i-k} - \mu_{\sigma(i)} - i + \sigma(i)} \right) t^{kn+|\lambda|-|\mu|}.$$

Since the sum over  $\sigma \in S_{k+r}$  is finite, we only need to show that  $R_\sigma$  is D-finite. Indeed using the Hadamard product on the variable  $t$ :

$$\begin{aligned} R_\sigma &= \left( \prod_{i=k+1}^{k+r} h_{\lambda_{i-k} - \mu_{\sigma(i)} - i + \sigma(i)} t^{|\lambda|-|\mu|} \right) \sum_n \left( \prod_{i=1}^k h_{n - \mu_{\sigma(i)} - i + \sigma(i)} \right) t^{kn} \\ &= \left( \prod_{i=k+1}^{k+r} h_{\lambda_{i-k} - \mu_{\sigma(i)} - i + \sigma(i)} t^{|\lambda|-|\mu|} \right) \cdot \bigodot_{i=1}^k \left( \sum_n h_{n - \mu_{\sigma(i)} - i + \sigma(i)} t^{kn} \right). \end{aligned}$$

which is a D-finite symmetric function by the same argument as Lemma 4.3. ♣

This last theorem will play an important role in Chapter 5, where we find generating functions of SSYT restricted to specific families of shapes.

## Chapter 5

# D-finiteness and the scalar and Kronecker products

In this chapter we present a few closure properties of D-finiteness under the scalar and Kronecker products of symmetric functions. We study some identities that suggest the possibility of an extension of these closure properties.

### 5.1 Kronecker product

Recall that the Kronecker product of symmetric functions is given by  $p_\lambda * p_\mu = \delta_{\lambda\mu} z_\lambda p_\lambda$ . In Gessel's treatise on D-finite symmetric functions, most of the results stem from the following:

**Theorem 5.1** (Gessel [7]). *If  $f$  and  $g$  are symmetric functions that are D-finite in the  $p_i$ 's (and maybe in some other variable  $t$ ), then  $f * g$  is D-finite in these variables.*

*Proof.* Let  $f = \sum_\lambda a_\lambda p_\lambda$  and  $g = \sum_\mu b_\mu p_\mu$  be symmetric functions and set  $v = \sum_\gamma z_\gamma p_\gamma$ . We have that

$$f * g = \sum_\lambda a_\lambda b_\lambda z_\lambda p_\lambda = f \odot g \odot v.$$

On the other hand, recalling that  $z_\lambda = 1^{r_1} r_1! 2^{r_2} r_2! \cdots$  we can write  $v$  as follows:

$$\begin{aligned} v &= \sum_{r_1, r_2, \dots} 1^{r_1} r_1! 2^{r_2} r_2! \cdots p_1^{r_1} p_2^{r_2} \cdots \\ &= \left( \sum_{r_1} r_1! (1p_1)^{r_1} \right) \left( \sum_{r_2} r_2! (2p_2)^{r_2} \right) \cdots \\ &= A(1p_1)A(2p_2) \cdots . \end{aligned}$$

where  $A(y) = \sum_{n \geq 0} n!y^n$  is a D-finite generating function since its coefficients are P-recursive. Thus  $v$  is D-finite by using part 1 of Theorem 4.1. We have obtained that  $f$ ,  $g$  and  $v$  are D-finite. Hence Theorem 4.1 provides the desired result.  $\clubsuit$

### 5.1.1 Some Kronecker product identities

In this subsection we present some useful identities involving Kronecker products.

**Lemma 5.1** (Mishna [16]). *If  $f$  and  $g$  are two symmetric functions such that  $f(p_1, p_2, \dots) = \prod_{i \geq 1} a_i(p_i)$  and  $g(p_1, p_2, \dots) = \prod_{j \geq 1} c_j(p_j)$ . Then  $f * g = \prod_{m \geq 1} a_m(p_m) * c_m(p_m)$ .*

*Proof.* We write  $a_i(p_i) = \sum_{k \geq 0} a_{ik} p_i^k$  and  $c_i(p_i) = \sum_{k \geq 0} c_{ik} p_i^k$ ,

$$\begin{aligned}
 f * g &= \left( \prod_{i \geq 1} a_i(p_i) \right) * \left( \prod_{j \geq 1} c_j(p_j) \right) \\
 &= \left( \sum_{k_1, k_2, \dots \geq 0} a_{1k_1} a_{2k_2} \cdots p_1^{k_1} p_2^{k_2} \cdots \right) * \left( \sum_{k_1, k_2, \dots \geq 0} c_{1k_1} c_{2k_2} \cdots p_1^{k_1} p_2^{k_2} \cdots \right) \\
 &= \sum_{k_1, k_2, \dots \geq 0} z_1^{k_1} z_2^{k_2} \cdots a_{1k_1} c_{1k_1} a_{2k_2} c_{2k_2} \cdots p_1^{k_1} p_2^{k_2} \cdots \\
 &= \prod_{i \geq 1} \sum_{k \geq 0} k! i^k a_{ik} c_{ik} p_i^k \\
 &= \prod_{i \geq 1} a_i(p_i) * c_i(p_i).
 \end{aligned}$$

$\clubsuit$

As stated in [16], this lemma in combination with the algorithms of Chyzak, Mishna and Salvy [5] (which are implemented in Maple) calculate the differential equations satisfied by the Kronecker product  $f * g$  (for  $f$  and  $g$  as above) when the ones satisfied by  $f$  and  $g$  are known. Their algorithm computes Grobner basis in the Weyl algebra, and a description is beyond the scope of this thesis.

The following is an example of a symmetric function which can be decomposed as in Lemma 5.1:

**Example 5.1.** Consider

$$h[e_2] = \exp \left( \sum_{k \geq 1} \frac{p_k[e_2]}{k} \right).$$

We can write this symmetric function as a product of functions depending on only one of the  $p_i$ 's. Indeed:

$$\begin{aligned} h[e_2] &= \exp \left( \sum_{k \geq 1} \frac{p_k}{k} \left[ \frac{p_1^2}{2} - \frac{p_2}{2} \right] \right) \\ &= \exp \left( \sum_{k \geq 1} \frac{p_k^2}{2k} - \frac{p_{2k}}{2k} \right) \\ &= \prod_{k \geq 1} \exp \left( \frac{p_k^2}{2k} - \frac{p_{2k}}{2k} \right) \\ &= \prod_{k \text{ even}} \exp \left( \frac{p_k^2}{2k} - \frac{p_k}{k} \right) \prod_{k \text{ odd}} \exp \left( \frac{p_k^2}{2k} \right). \end{aligned}$$

More generally, if  $g$  is a sum of symmetric functions, each depending on only one of the  $p_i$ 's, then the plethysms  $h[g]$  and  $e[g]$  can be written as a product of functions, each depending on only one of the  $p_i$ 's. Indeed, if  $g = \sum_{k \geq 1} g_k(p_k)$ , then

$$\begin{aligned} h[g] &= h \left[ \sum_{k \geq 1} g_k(p_k) \right] \\ &= \exp \left( \sum_{i \geq 1} \frac{p_i \left[ \sum_{k \geq 1} g_k(p_k) \right]}{i} \right) \\ &= \exp \left( \sum_{i \geq 1} \frac{\sum_{k \geq 1} g_k(p_{ki})}{i} \right) \\ &= \exp \left( \sum_{i, k \geq 1} \frac{g_k(p_{ki})}{i} \right), \end{aligned}$$

which gives the desired result by grouping similar terms in the exponent. The plethysm  $e[g]$  is verified in a similar way.

The following identities (Macdonald [14]) are satisfied:

$$\begin{aligned}
h(t) &= \sum_n h_n t^n = \exp\left(\sum_n \frac{p_n t^n}{n}\right), \\
e(t) &= \sum_n e_n t^n = \exp\left(\sum_n (-1)^{n-1} \frac{p_n t^n}{n}\right), \\
s(t) &= \sum_\lambda s_\lambda t^{|\lambda|} = \exp\left(\sum_n \frac{p_n^2 t^{2n}}{2n} + \frac{p_{2n-1} t^{2n-1}}{2n-1}\right), \\
s(t)e(t)^{-1} &= \sum_{\lambda \in Ev} s_\lambda t^{|\lambda|} = \exp\left(\sum_n \frac{p_n^2 t^{2n}}{2n} + \frac{p_{2n-1} t^{2n-1}}{2n-1} + \frac{(-1)^n p_n t^n}{n}\right), \\
s(t)h(t)^{-1} &= \sum_{\lambda' \in Ev} s_{\lambda'} t^{|\lambda'|} = \exp\left(\sum_n \frac{p_n^2 t^{2n}}{2n} + \frac{p_{2n-1} t^{2n-1}}{2n-1} - \frac{p_n t^n}{n}\right).
\end{aligned}$$

where  $Ev := \{\lambda : \text{all parts of } \lambda \text{ are even}\}$ . Also, with the notation  $s = s(1)$ ,  $se^{-1} = s(1)e(1)^{-1}$  and  $sh^{-1} = s(1)h(1)^{-1}$ , we have the following result:

**Theorem 5.2** (Kronecker product identities, Mishna [16]). *The identities from Table 5.1 are satisfied, where*

$$\begin{aligned}
m_{\text{odd}(\text{even})}(t) &= \exp\left(\sum_{n \text{ odd}(\text{even})} \frac{p_n t^n}{n(1 - p_n t^n)}\right), \\
q(t) &= \exp\left(\sum_n \frac{p_n^2 t^{2n}}{2n(1 - p_n^2 t^{2n})}\right), \\
j(t) &= \exp\left(\sum_{n \text{ even}} \frac{p_n t^n}{n(1 + p_n t^n)}\right), \\
l(t) &= \prod_{n \geq 1} (1 - p_n^2 t^{2n})^{-1/2}.
\end{aligned}$$

*Proof.* The first line of the table is obtained by using the fact that  $h = h(1)$  is the identity with respect to the Kronecker product (Equation (3.9)). The rest of the identities are proved by using Lemma 5.1 and algorithms from [5]. As in [16], we present a typical argument for

*	$h(t)$	$e(t)$	$s(t)$	$s(t)h(t)^{-1}$	$s(t)e(t)^{-1}$
$h$	$h(t)$	$e(t)$	$s(t)$	$s(t)h(t)^{-1}$	$s(t)e(t)^{-1}$
$e$		$h(t)$	$s(t)$	$s(t)e(t)^{-1}$	$s(t)h(t)^{-1}$
$s$			$l(t)m_{odd}(t)$	$l(t)q(t)$	$l(t)q(t)$
$sh^{-1}$				$l(t)m_{even}(t)$	$l(t)j(t)$
$se^{-1}$					$l(t)m_{even}(t)$

Table 5.1: A family of Kronecker product identities

one of the products that can be adapted to show the remaining ones;

$$\begin{aligned}
 e * sh^{-1} &= \prod_k \exp\left((-1)^{k-1} \frac{p_k}{k}\right) * \prod_l \exp\left(\frac{p_l^2}{2l} + \frac{p_{2l-1}}{2l-1} - \frac{p_l}{l}\right) \\
 &= \prod_{k \text{ odd}} \exp\left(\frac{p_k}{k}\right) * \exp\left(\frac{p_k^2}{2k} + \frac{p_k}{k} - \frac{p_k}{k}\right) \\
 &\quad \cdot \prod_{k \text{ even}} \exp\left(-\frac{p_k}{k}\right) * \exp\left(\frac{p_k^2}{2k} - \frac{p_k}{k}\right), \text{ by Lemma 5.1.} \\
 &= \prod_{k \text{ odd}} g(p_k) \prod_{k \text{ even}} f(p_k),
 \end{aligned}$$

where  $g(p_k) = g_1(p_k) * g_2(p_k)$ ,  $f(p_k) = f_1(p_k) * f_2(p_k)$  and  $g_1, g_2, f_1, f_2$  are the exponentials above. Then using Maple (algorithms in [5]), we get the differential equations satisfied by each of these Kronecker products,

$$\begin{cases} -p_k g(p_k) - k \frac{\partial(g(p_k))}{\partial p_k} = 0 \text{ for } k \text{ odd,} \\ (p_k + 1) f(p_k) - k \frac{\partial(f(p_k))}{\partial p_k} = 0 \text{ for } k \text{ even.} \end{cases}$$

These equations can be solved in Maple, yielding:

$$e * sh^{-1} = \prod_{k \text{ odd}} \exp\left(\frac{p_k^2}{2k}\right) \prod_{k \text{ even}} \exp\left(\frac{p_k^2}{2k} + \frac{p_k}{k}\right) = se^{-1}.$$

We have that  $e * sh^{-1} = se^{-1}$  and we wish to obtain the more general equality  $e * s(t)h(t)^{-1} = s(t)e(t)^{-1}$ . In general if  $u, v$  and  $w$  are symmetric functions with  $u * v = w$ , then  $u * v(t) =$

$w(t)$  where  $v(t)$  and  $w(t)$  denote the replacement  $p_k \rightarrow t^k p_k$  in  $v$  and  $w$  respectively. This can be easily shown by writing  $u = \sum_{\lambda} a_{\lambda} p_{\lambda}$ ,  $v = \sum_{\mu} b_{\mu} p_{\mu}$ . In the particular case  $u = e$ ,  $v = sh^{-1}$  we obtain

$$e * s(t)h(t)^{-1} = s(t)e(t)^{-1}$$

as desired. ♣

## 5.2 Scalar product

Concerning the closure properties of the scalar product, Gessel proved the following result:

**Theorem 5.3** (Scalar Product, Gessel[7]). *If  $f$  and  $g$  are symmetric functions such that:*

1.  $f$  and  $g$  are  $D$ -finite with respect to the  $p_i$ 's and another variable  $t$ .
2.  $g$  involves only a finite number of  $p_i$ 's.
3.  $\langle f, g \rangle$  is well-defined as a formal power series in  $t$ .

Then

$$\langle f, g \rangle \text{ is } D\text{-finite with respect to } t.$$

*Proof.* Using Theorem 5.1, we know that  $f * g = u(t, p_1, p_2, \dots)$  is  $D$ -finite. Also notice that this Kronecker product involves only a finite number of  $p_i$ 's since  $g$  does. Write  $f * g = u(t, p_1, p_2, \dots, p_m)$  for some  $m \in \mathbb{N}$ . Setting  $p_i = 1$  for all  $i$ :

$$f * g|_{p_i=1} = u(t, 1, 1, \dots, 1) = \langle f, g \rangle$$

By using part 2 of Theorem 4.2, we get the desired result. ♣

**Remark 5.1.** It is worth pointing out that in general by setting  $p_i = 1$  for a finite number of values of  $i$ ,  $D$ -finiteness is preserved. This is a clear consequence of Theorem 4.2.

Notice that the conditions of Theorem 5.3 are sufficient but they are not all necessary.

Indeed,

$$\begin{aligned}
\langle h, h(t) \rangle &= h * h(t)|_{p_i=1} \\
&= h(t)|_{p_i=1} \\
&= \exp\left(\sum_n \frac{p_n t^n}{n}\right)\Big|_{p_i=1} \\
&= \exp\left(\sum_n \frac{t^n}{n}\right) \\
&= \exp\left(\log\left(\frac{1}{1-t}\right)\right) \\
&= \frac{1}{1-t},
\end{aligned}$$

which is a rational function of  $t$  and so it is D-finite. However, neither of the symmetric functions in the scalar product involves a finite number of variables, that is, the second condition of Theorem 5.3 is not satisfied. This suggests the idea that these conditions may be relaxed.

It is not always true that the scalar product of two symmetric series that are D-finite in the  $p_i$ 's and  $t$  is a D-finite power series. For example, let  $d(n)$  be any non-P-recursive sequence. Notice that  $\sum_n d(n)p_n$  and  $\sum_m \frac{p_m t^m}{m}$  are D-finite, but  $\sum_n d(n)t^n = \langle \sum_n d(n)p_n, \sum_m \frac{p_m t^m}{m} \rangle$  is not D-finite.

Now we present a consequence of Theorem 5.5 given by Gessel, that can be applied to enumerative problems. We use this result in Sections 5.3 and 5.4.

**Corollary 5.4** (Gessel). *Let  $f$  be a D-finite symmetric function and let  $\mathcal{B}$  be a finite set of positive integers. Define integers  $b_n$  as follows:  $b_n$  is the sum over all tuples  $(\lambda_1, \dots, \lambda_n) \in \mathcal{B}^n$  of the coefficient of  $x_1^{\lambda_1} \dots x_n^{\lambda_n}$  in  $f$ . Then  $B(t) = \sum_{n \geq 0} b_n t^n$  is D-finite.*



*Proof.* By definition we have that  $b_n = \sum_{\lambda \in \mathcal{B}^n} \langle f, h_\lambda \rangle$ . Therefore,

$$\begin{aligned} \sum_{n \geq 0} b_n t^n &= \sum_{n \geq 0} \left\langle f, \sum_{\lambda \in \mathcal{B}^n} h_\lambda \right\rangle t^n \\ &= \left\langle f, \sum_{n \geq 0} \sum_{\lambda \in \mathcal{B}^n} h_\lambda t^n \right\rangle \\ &= \left\langle f, \sum_{n \geq 0} \left( \sum_{i \in \mathcal{B}} h_i t \right)^n \right\rangle \\ &= \left\langle f, \left( 1 - t \sum_{i \in \mathcal{B}} h_i \right)^{-1} \right\rangle. \end{aligned}$$

And the result follows from Theorem 5.3. ♣

### 5.2.1 Extension of closure properties for D-finite scalar products

In Theorem 5.3 the finiteness condition over the number of  $p_i$ 's involved in  $g$  is too restrictive. There are many examples of D-finite scalar products not satisfying this condition. The following theorem involves some pairs of symmetric functions that do not satisfy the second condition of Theorem 5.3 but their scalar product is still D-finite.

**Theorem 5.5.** *The identities from Table 5.2 are satisfied.*

*Proof.* For the first two lines of this table, it suffices to make the replacement  $p_i = 1$  in the first two lines of Theorem 5.2. For the last three lines we do not make this substitution, as the sums become too complicated. Instead we use the orthogonality of the Schur functions. For instance,

$$\begin{aligned} \langle s, s(t) \rangle &= \left\langle \sum_{\lambda} s_{\lambda}, \sum_{\mu} s_{\mu} t^{|\mu|} \right\rangle \\ &= \sum_{\lambda, \mu} \langle s_{\lambda}, s_{\mu} \rangle t^{|\mu|} \\ &= \sum_{\lambda, \mu} \delta_{\lambda, \mu} t^{|\mu|} \\ &= \prod_k \frac{1}{1 - t^k}. \end{aligned}$$

Similarly for the other products. ♣

$\langle, \rangle$	$h(t)$	$e(t)$	$s(t)$	$s(t)h(t)^{-1}$	$s(t)e(t)^{-1}$
$h$	$\frac{1}{1-t}$	$1+t$	$\frac{1}{1-t}$	$1$	$\frac{1}{1-t^2}$
$e$		$\frac{1}{1-t}$	$\frac{1}{1-t}$	$\frac{1}{1-t^2}$	$1$
$s$			$\prod_{k \geq 0} \frac{1}{(1-t^k)}$	$\prod_{k \geq 0} \frac{1}{(1-t^{2k})}$	$\prod_{k \geq 0} \frac{1}{(1-t^{2k})}$
$sh^{-1}$				$\prod_{k \geq 0} \frac{1}{(1-t^{2k})}$	$\prod_{k \geq 0} \frac{1}{(1-t^{4k})}$
$se^{-1}$					$\prod_{k \geq 0} \frac{1}{(1-t^{2k})}$

Table 5.2: A family of scalar product identities.

**Remark 5.2.** Notice that the scalar products in the first two lines are all D-finite. However, the remaining ones are not, following Theorem 4.3 on the generating function of partitions.

The first two lines of Table 5.2 may suggest that in fact we can extend Gessel’s result by relaxing the conditions for a scalar product to be D-finite and the last three suggest care is needed. As a first step towards this goal, we prove in the next chapter:

**Theorem 6.8.** Let  $f, g$  be polynomials in the  $p_i$ ’s and possibly another variable  $t$ . Then  $\langle hf, h(t)g \rangle$  is D-finite as a function of  $t$ .

### 5.3 Applications to tableaux enumeration

Recall from Section 3.5.3 the scalar product

$$Y(t) = \sum_n y_n t^n = \left\langle \sum_\lambda s_\lambda, \sum_n h_1^n \right\rangle,$$

where  $y_n$  represents the number of SYT of size  $n$ . Since

$$\sum_\lambda s_\lambda = h[e_1 + e_2]$$

is D-finite as shown in Section 4.4, and by Corollary 5.4 we have that  $Y(t)$  is D-finite. Similarly, if for some family  $\mathcal{F}$  of partitions it is true that

$$\sum_{\lambda \in \mathcal{F}} s_{\lambda}$$

is D-finite, then as a consequence of Corollary 5.4, the ordinary generating function of the number of SSYT of shape  $\lambda \in \mathcal{F}$  and type of the form  $(j^n)$  ( $j$  fixed) is D-finite. More generally, for any fixed partition  $\mu$ , if

$$\sum_{\lambda \in \mathcal{F}} s_{\lambda/\mu}$$

is D-finite, then the ordinary generating function for the number of SSYT of shape  $\lambda/\mu$  for  $\lambda \in \mathcal{F}$  and type of the form  $(j^n)$  is also D-finite. When  $\mathcal{F}$  is the family of all partitions, we have:

$$\sum_{\lambda} s_{\lambda/\mu} = h[e_1 + e_2] \sum_{\lambda} s_{\mu/\lambda}$$

which is D-finite, since  $h[e_1 + e_2]$  is D-finite and the second sum over  $\lambda$  is a finite sum of Schur functions ( $\lambda$  is bounded by  $\mu$ ) so it is a polynomial in the  $p_i$ 's. A more restrictive family  $\mathcal{F}$  is reviewed in the following example:

**Example 5.2.** Let  $r$  be a fixed integer, for  $\mathcal{F} = \{\lambda : \lambda \text{ has at most } r \text{ rows}\}$  (known as the partitions of bounded height) and a fixed partition  $\mu$ , Gessel proved that the symmetric series

$$B_r(\mu) = \sum_{\lambda \in \mathcal{F}} s_{\lambda/\mu}$$

is D-finite. In 1968, Gordon and Houten [9] and Bender and Knuth [3] had given a formula for this series, and two years later Gordon [8] published a simplification of this formula. Gessel connected this results with his results for the case  $\mu = (0)$ . However, we do not present these formulas here, as they are beyond the scope of our current research.

On the other hand, recall from Chapter 4 that the series  $\sum_n s_{(n^k, \lambda)/\mu} t^{nk+|\lambda|-|\mu|}$  is D-finite and so  $\sum_n s_{(n^k, \lambda)/\mu}$  is D-finite as well, since it results from setting  $t = 1$  in the first sum. Being consistent with the notation above, this sum corresponds to the family  $\mathcal{F}$  of partitions of the form  $(n^k, \lambda)$ . Notice that only a finite number of Schur functions in that sum are indexed by compositions that are not partitions, and that adding or subtracting any finite sum of Schur functions has no effect over the D-finiteness of a symmetric function. As a consequence of Corollary 5.4, the following result arises,

**Corollary 5.6.** *The number of SSYT of type  $(j^n)$  and shape  $(m^k, \lambda)/\mu$  is a P-recursive function of  $n$ , for  $\lambda, \mu, j$  and  $k$  fixed.*

In particular the number of SYT of shape  $(m^k, \lambda)/\mu$  is a P-recursive function of  $m$ . We are interested in finding a closed formula for the ordinary generating function

$$\left\langle \sum_m s_{m^k, \lambda}, \sum_n h_j^n t^n \right\rangle$$

of the SSYT mentioned in Corollary 5.6. We can obtain a general formula for the recurrence satisfied by its coefficients in the case  $j = 1$ , as a direct consequence of the hook formula, in the same manner we did for the case  $k = 1, j = 1$  in Section 4.1.

We believe that it is not always the case that any sum of  $s_\lambda$ 's (for a particular family of partitions  $\lambda$ ) is D-finite. Consider the following problems.

**Problem 5.1.** *Are the symmetric series  $\sum_n s_{n, n-1, n-2, \dots, 1}$  and  $\sum_n s_n$  D-finite with respect to the  $p_i$ 's?*

**Problem 5.2.** *What are some sufficient conditions for the sum of all Schur functions indexed by the skew partitions in a given family  $\mathcal{F}$  to be D-finite? Notice that this can be applied to tableaux enumeration (as in Section 5.3) by showing the P-recursive nature with respect to  $n$  of the number of SSYT of type  $j^n$  and shape in  $\mathcal{F}$ , for any fixed  $j$ .*

## 5.4 Other applications

In 1986, Goulden and Jackson [10] introduced a solution to the problem of counting a particular class of graphs using Grobner bases. Four years later, Gessel solved the same problem in a simpler way by using symmetric functions. We present here a brief summary of his work.

Consider the combinatorial structure of labeled graphs with vertices  $1, 2, \dots, n$ . Define the weight of one such graph as the monomial on  $x_1, x_2, \dots, x_n$  where the exponent of  $x_i$  is the degree of the vertex  $i$  in the graph. Hence if we consider the infinite formal sum of all these weights for a given set of graphs, the coefficient of the monomial  $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}$  represents the number of graphs in that set where the degree of the vertex  $i$  is  $\lambda_i$ .

Recall the D-finite plethysms considered in Sections 3.5.1 and 4.4. Each of these symmetric functions represents the generating function (sum of all the weights) of a different class of graphs, as presented in Table 5.3.

Class of labeled graphs.	Formula for the generating function.
Graphs with multiple edges and loops (a loop contributes 2 to the degree of its vertex).	$h[h_2] = \prod_{i \leq j} (1 - x_i x_j)^{-1}.$
Graphs with multiple edges but no loops.	$h[e_2] = \prod_{i < j} (1 - x_i x_j)^{-1}.$
Graphs with loops but no multiple edges.	$e[h_2] = \prod_{i \leq j} (1 + x_i x_j).$
Graphs with no multiple edges and no loops (simple labeled graphs).	$e[e_2] = \prod_{i < j} (1 + x_i x_j).$

Table 5.3: Classes of labelled graphs.

The following is a direct consequence of Corollary 5.4,

**Corollary 5.7** (Gessel [7]). *The number of graphs in any of the classes above, on an  $n$ -element set, such that all the degrees are in a given finite set  $\mathcal{B}$ , is P-recursive.*

Notice that the case  $\mathcal{B} = \{j\}$  in Corollary 5.7 allows us to show that the number of  $j$ -regular graphs on  $n$  vertices, which is the coefficient of  $(x_1 x_2 \cdots x_n)^j$  in each of the symmetric functions above, is P-recursive in  $n$ .

Since any symmetric function can be written as a linear combination of the monomial basis, the scalar product may be used to get coefficients of a particular monomial in a given symmetric function as well. Gessel [7] worked on some enumerative results such as counting alternating permutations and increasing support sequences, all of which involved coefficient extraction using the scalar product.

In some cases it is not easy to find a formula for the number of combinatorial structures of certain type and size. However, if we can find a closed form for its generating function, we have all the information about its coefficients as well. In his paper, Gessel [7] provided a simple way using symmetric functions to obtain the generating functions of certain

combinatorial structures, such as partitions of multisets and nonnegative integer matrices.

## Chapter 6

# Reduced Kronecker product

In 1938, Murnaghan [17] introduced the concept of reduced notation for Schur functions. In 1991 Thibon [24] introduced a more general approach to reduced notation of symmetric functions using the theory of Hopf algebras. Thibon's work has found several applications to areas such as quantum electronics and molecular science. One of our most important contributions consists establishing a connection between Thibon's work and the notion of D-finiteness. In this chapter, we summarize Murnaghan's and Thibon's results and connect them with the theory of D-finite symmetric functions.

### 6.1 Reduced Kronecker coefficients

The coefficients  $C'_{\lambda\mu}$  that arise when the product  $s_\lambda * s_\mu$  is expressed in terms of Schur function, are known as *Kronecker coefficients*. No one has been able to find a general combinatorial or algebraic formula for these coefficients. Murnaghan showed the stability of the coefficients appearing in the Kronecker products between certain families of Schur functions, and he called these coefficients the *reduced Kronecker coefficients* ( $\overline{C}'_{\lambda\mu}$ ). These coefficients were studied by Briand, Orellana and Rosas [4] as a way to gain information about the Kronecker coefficients. Define  $\lambda[n] = (n - |\lambda|, \lambda_1, \lambda_2, \dots)$  (for a partition  $\lambda = (\lambda_1, \lambda_2, \dots)$ ), which is a partition only if  $n \geq |\lambda| + \lambda_1$ .

**Theorem 6.1** (Murnaghan [18]). *There is a family of non-negative integers  $(\overline{C}'_{\lambda\mu})$ , indexed by triplets of partitions  $(\lambda, \mu, \gamma)$  such that only a finite number of terms  $\overline{C}'_{\lambda\mu}$  are nonzero for  $\lambda, \mu$  fixed, and for all  $n \geq 0$  sufficiently large:*

$$s_{\lambda[n]} * s_{\mu[n]} = \sum_{\gamma} \overline{C}_{\lambda\mu}^{\gamma} s_{\gamma[n]}.$$

Moreover, the coefficient  $\overline{C}_{\lambda\mu}^{\gamma}$  is nonzero only when all the inequalities:  $|\lambda| \leq |\mu| + |\gamma|$ ,  $|\mu| \leq |\lambda| + |\gamma|$  and  $|\gamma| \leq |\lambda| + |\mu|$  are satisfied.

In the case where  $\lambda[n]$  is not a partition, the Schur function  $s_{\lambda[n]}$  can always be written in terms of Schur functions indexed by partitions using the Jacobi-Trudi identity. The objects  $s_{\bullet,\lambda} = s_{\lambda[n]}$  are often called *reduced Schur functions*.

**Example 6.1.** Consider the partitions  $\lambda = (2)$  and  $\mu = (1, 1)$ . Using the package [23] in Maple, we get the following (for simplicity, we omit the brackets from the partitions indexing Schur functions):

$$\begin{aligned} \text{For } n = 5 : \quad & s_{3,2} * s_{3,1,1} = s_{4,1} + s_{3,2} + 2s_{3,1,1} + s_{2,2,1} + s_{2,1,1,1} \\ n = 6 : \quad & s_{4,2} * s_{4,1,1} = s_{5,1} + s_{4,2} + 2s_{4,1,1} + s_{3,3} + 2s_{3,2,1} + s_{3,1,1,1} + s_{2,2,1,1} \\ n = 7 : \quad & s_{5,2} * s_{5,1,1} = s_{6,1} + s_{5,2} + 2s_{5,1,1} + s_{4,3} + 2s_{4,2,1} + s_{4,1,1,1} + s_{3,3,1} + s_{3,2,1,1} \\ n = 8 : \quad & s_{6,2} * s_{6,1,1} = s_{7,1} + s_{6,2} + 2s_{6,1,1} + s_{5,3} + 2s_{5,2,1} + s_{5,1,1,1} + s_{4,3,1} + s_{4,2,1,1} \\ n = 9 : \quad & s_{7,2} * s_{7,1,1} = s_{8,1} + s_{7,2} + 2s_{7,1,1} + s_{6,3} + 2s_{6,2,1} + s_{6,1,1,1} + s_{5,3,1} + s_{5,2,1,1} \end{aligned}$$

We can see that for  $n \geq 7$ , the reduced Kronecker coefficients stabilize. We have, for example,  $\overline{C}_{(2),(1,1)}^{(3)} = 1$  and  $\overline{C}_{(2),(1,1)}^{(2,1)} = 2$ . We can deduce that:

$$n \geq 7; \quad s_{\bullet,2} * s_{\bullet,1,1} = s_{\bullet,1} + s_{\bullet,2} + 2s_{\bullet,1,1} + s_{\bullet,3} + 2s_{\bullet,2,1} + s_{\bullet,1,1,1} + s_{\bullet,3,1} + s_{\bullet,2,1,1}.$$

There exists a general formula for the number  $n$  at which the Kronecker product  $s_{\bullet,\lambda} * s_{\bullet,\mu} = s_{\lambda[n]} * s_{\mu[n]}$  stabilizes and it was given by Briand, Orellana and Rosas in [4]. For a fixed positive integer  $n$ , consider the linear operator given by  $\overline{s_{\gamma}} = s_{\gamma[n]}$ .

**Example 6.2.** If  $n = 10$ , then  $\overline{s_4 + s_{4,2,1}} = s_{6,5} + s_{2,4,3,1}$ , where  $(6, 5)$  is a partition, but  $(2, 4, 3, 1)$  is not. By applying the Jacobi-Trudi identity we obtain that  $s_{2,4,2,1} = -s_{3,3,3,1}$ . Thus  $\overline{s_4 + s_{4,2,1}} = s_{6,5} - s_{3,3,3,1}$ .

Littlewood found the following general formula for the Kronecker product of two reduced Schur functions.



**Theorem 6.2** (Littlewood[13]). *For two fixed partitions  $\lambda, \mu$  with  $|\lambda| = |\mu|$  and a fixed nonnegative integer  $n \geq 0$ :*

$$s_{\lambda[n]} * s_{\mu[n]} = \sum_{\alpha, \beta, \gamma} \overline{s_{\lambda/\alpha\gamma} s_{\mu/\beta\gamma} (s_{\alpha} * s_{\beta})}.$$

where  $\alpha\gamma = (\alpha_1 + \gamma_1, \alpha_2 + \gamma_2, \dots)$ , the sum is over all partitions  $\alpha, \beta$  and  $\gamma$  with  $|\alpha| = |\beta|$ , and  $\overline{s_{\theta}} = s_{\theta[n]}$  for any partition  $\theta$ .

**Example 6.3.** Let us illustrate Theorem 6.2 for  $\lambda = 3, \mu = 2$ . Considering all the possible partitions  $\alpha, \beta, \gamma$  we obtain:

For $\alpha = \beta = \gamma = 0$	$s_3 s_2$
For $\alpha = \beta = 0$ and $\gamma = 1$	$s_2 s_1 (s_0 * s_0) = s_2 s_1.$
For $\alpha = \beta = 0$ and $\gamma = 2$	$s_1 s_0 (s_0 * s_0) = s_1$
For $\alpha = \beta = 1$ and $\gamma = 0$	$s_2 s_1 (s_1 * s_1) = s_2 s_1 s_1.$
For $\alpha = \beta = 1$ and $\gamma = 1$	$s_1 s_0 (s_1 * s_1) = s_1 s_1.$
For $\alpha = \beta = 2$ , where necessarily $\gamma = 0$ ,	$s_1 s_0 (s_2 * s_2) = s_1 s_2.$

Thus, adding up all these terms,

$$s_{3[n]} * s_{2[n]} = \overline{s_3 s_2 + s_1 s_2 + s_1 + s_1 s_1 s_2 + s_1 s_1 + s_1 s_2},$$

expanding the products,

$$s_{3[n]} * s_{2[n]} = \overline{s_5 + s_{4,1} + s_{3,2} + s_1 + s_4 + 2s_{3,1} + s_{2,2} + s_{2,1,1} + s_2 + s_{1,1} + 2s_3 + 2s_{2,1}}.$$

Hence for  $n=6$ , by linearity of the over line operator,

$$\begin{aligned} s_{3[6]} * s_{2[6]} &= s_{1,5} + s_{1,4,1} + s_{1,3,2} + s_{5,1} + s_{2,4} + 2s_{2,3,1} + s_{2,2,2} + s_{2,2,1,1} + s_{4,2} + s_{4,1,1} \\ &\quad + 2s_{3,3} + 2s_{3,2,1}, \end{aligned}$$

Now using the Jacobi-Trudi identity,

$$\begin{aligned} s_{3[6]} * s_{2[6]} &= -s_{4,2} - s_{3,2,1} - s_{2,2,2} + s_{5,1} - s_{3,3} + s_{2,2,2} + s_{2,2,1,1} + s_{4,2} + s_{4,1,1} \\ &\quad + 2s_{3,3} + 2s_{3,2,1} \\ &= s_{5,1} + s_{4,1,1} + s_{3,3} + s_{3,2,1} + s_{2,2,1,1}. \end{aligned}$$

Partition $\lambda$	Reduced Kronecker coefficients
$\lambda = (m)$	$\bar{C}_{k,l}^\lambda = \frac{1}{2}(l - k + m + 2)$ for $k > m$ . $\frac{1}{2}(l + k - m + 2)$ for $m \geq k$ .
$\lambda = (m, 1)$	$\bar{C}_{k,l}^\lambda = l - k + m + 1$ for $k > m$ . $l + k - m$ for $m \geq k$ .
$\lambda = (m, 1^2)$	$\bar{C}_{k,l}^\lambda = \frac{1}{2}(l - k + m + 1)$ for $k > m + 1$ . $\frac{1}{2}(l + k - m + 2)$ for $m \geq k - 1$ .
$\lambda = (m, l)$	Algorithm.
$\lambda = (m, l, n)$	Algorithm.

Table 6.1: Coefficients for two-row shape Schur functions

Some work has been done to characterize the coefficients that appear in the Kronecker product of reduced Schur functions indexed by two-row partitions. In particular:

**Corollary 6.3** (Two-row shape partitions, Scharf, Thibon and Wybourne [20]). *For  $k \geq l$  fixed:*

$$s_{k[n]} * s_{l[n]} = \sum_{\lambda} \bar{C}_{k,l}^\lambda s_{\lambda[n]} = \sum_{p=0}^l \sum_{q=0}^p s_{k-p} s_{l-p} s_{p-q}.$$

Scharf, Thibon and Wybourne gave explicit formulas and algorithms to calculate the coefficients that appear in this case by noticing that  $\lambda$  can have at most three rows, which results from the Littlewood-Richardson multiplication rule of Schur functions. We summarize their results in Table 6.1.

Other authors such as Rosas and Remmel have extended these results from combinatorial and algebraic points of view. However, there is no existing work relating them to the notion of D-finiteness.

**Problem 6.1.** *Can we explain the Kronecker product of two-row shape Schur functions (see Scharf, Thibon and Wybourne [20]) using D-finiteness, more specifically by using the algorithms by Mishna [15] and Chyzak, Mishna and Salvy [5]?*

## 6.2 Adjoint multiplication

The adjoint operator to multiplication with respect to the scalar product has a very nice characterization, and is important in the study of reduced Kronecker products. Recall that

$\Lambda$  denotes the space of all symmetric functions and consider  $f \in \Lambda$ . Define the *adjoint multiplication* of  $f$  as the homomorphism  $D_f : \Lambda \rightarrow \Lambda$ , such that for all  $g_1, g_2 \in \Lambda$ :

$$\langle D_f g_1, g_2 \rangle = \langle g_1, f g_2 \rangle.$$

It is not immediately obvious that this defines the operator  $D_f$  uniquely. For this notice that the coefficient of  $m_\mu$  in  $D_f p_\lambda$  is given by  $\langle D_f p_\lambda, h_\mu \rangle = \langle p_\lambda, f h_\mu \rangle$ , so the expansion of  $D_f p_\lambda$  in terms of the monomial basis can be obtained entirely from the property above.

It is possible to find an explicit formula for the adjoint multiplication of each of the well known bases of the space  $\Lambda$  (see Macdonald [14]). Particularly, we are interested in the adjoint multiplication of  $s_\lambda$ , which is given by  $D_{s_\lambda} s_\mu = s_{\mu/\lambda}$  and the adjoint multiplication of the power symmetric functions  $p_n$ , which is  $D_{p_n}(f) = n \frac{\partial f}{\partial p_n}$ . We show this last equality, for which it suffices to prove it for  $f = p_\lambda$ . Consider two partitions  $\lambda$  and  $\mu$ , such that  $\mu = 1^{r_1} 2^{r_2} \dots n^{r_n} \dots$ . By definition we have,

$$\langle D_{p_n} p_\lambda, p_\mu \rangle = \langle p_\lambda, p_n p_\mu \rangle = \begin{cases} 0 & \text{if } \lambda \neq (1^{r_1} 2^{r_2} \dots n^{r_n+1} \dots). \\ z_\lambda & \text{if } \lambda = (1^{r_1} 2^{r_2} \dots n^{r_n+1} \dots). \end{cases}$$

Hence,  $D_{p_n} p_\lambda = z_\lambda \frac{p_\mu}{z_\mu}$  if  $\lambda = (1^{r_1} 2^{r_2} \dots n^{r_n+1} \dots)$ . Since  $z_\lambda = 1^{r_1} r_1! 2^{r_2} r_2! \dots n^{r_n+1} (r_n + 1)! \dots$  we have that  $\frac{z_\lambda}{z_\mu} = n(r_n + 1)$ . Therefore,

$$D_{p_n} p_\lambda = n(r_n + 1) p_\mu = n \frac{\partial p_\lambda}{\partial p_n} p_\lambda.$$

as wanted.

In general, for any  $f \in \Lambda$ , the adjoint multiplication of  $f$  amounts to a substitution:

$$D_f = f|_{p_i = i \frac{\partial}{\partial p_i}}.$$

This results from the equalities  $D_{fg} = D_f D_g$  and  $D_{f+g} = D_f + D_g$ , which are direct results of the definition of the adjoint multiplication.

**Example 6.4.** To clarify this definition and the properties above, consider the following examples:

- $D_{p_2}(e_5) = D_{p_2} \left( \frac{1}{5} p_5 - \frac{1}{4} p_1 p_4 - \frac{1}{6} p_3 p_2 + \frac{1}{6} p_3 p_1^2 + \frac{1}{8} p_1 p_2^2 - \frac{1}{12} p_2 p_1^3 + \frac{1}{120} p_1^5 \right)$   
 $= -\frac{1}{3} p_3 + \frac{1}{2} p_1 p_2 - \frac{1}{6} p_1^3.$
- $D_{s_{3,1}}(s_{4,3,2}) = s_{4,3,2/3,1}.$

- The symmetric function  $e = \sum_n e_n$  is equal to  $\exp\left(\sum_{k \geq 1} (-1)^{k-1} \frac{p_k}{k}\right)$ , from which we obtain

$$D_e = \exp\left(\sum_{k \geq 1} (-1)^{k-1} \frac{\partial}{\partial p_k}\right).$$

- Similarly for  $h = \sum_n h_n = \exp\left(\sum_{k \geq 1} \frac{p_k}{k}\right)$ :

$$D_h = \exp\left(\sum_{k \geq 1} \frac{\partial}{\partial p_k}\right).$$

### 6.3 Applying the adjoint multiplication

As we mentioned before, we are interested in weakening Gessel's conditions for the scalar product of two symmetric function to be D-finite. In order to obtain a fairly general example of a D-finite Kronecker product for which these conditions are not all satisfied, we need to recall one of the main results published by Thibon in 1991.

**Theorem 6.4** (Main theorem, Thibon [24]). *Let  $\{u_\lambda\}$ ,  $\{v_\lambda\}$  be adjoint bases<sup>1</sup> of  $\Lambda$  and  $f, g \in \Lambda$ , and let  $h = \sum_n h_n$ . Then,*

$$hf * hg = h\Phi,$$

where  $\Phi = \sum_{\lambda, \mu} (D_{v_\lambda} f)(D_{v_\mu} g)(u_\lambda * u_\mu)$  and the sum is taken over all partitions  $\lambda, \mu$ .

The proof of this theorem can be found in [24]. It uses some properties of the bialgebra structure provided by the Kronecker product of symmetric functions. Following the same steps from this proof, the following more general result can be shown:

**Theorem 6.5.** *Let  $\{u_\lambda\}$ ,  $\{v_\lambda\}$  be adjoint bases of  $\Lambda$  and  $f, g$  formal power series in the  $p_i$ 's and possibly another variable  $t$ . Then,*

$$hf * h(t)g = h(t)\Phi(t),$$

where  $\Phi(t) = \sum_{\lambda, \mu} (D_{v_\lambda} f)(D_{v_\mu} g)(u_\lambda * u_\mu)$  and  $h(t) = \sum_n h_n t^n$ .

---

<sup>1</sup>Recall that two bases  $\{u_\lambda\}$  and  $\{v_\lambda\}$  are said to be adjoint if  $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda\mu}$  for all partitions  $\lambda$  and  $\mu$ .

**Example 6.5.** Set  $f = g = p_k$ ,  $v_\lambda = p_\lambda$  and  $u_\lambda = \frac{p_\lambda}{z_\lambda}$  in Theorem 6.5,

$$\begin{aligned} hf * h(t)g = hp_k * h(t)p_k &= h(t) \sum_{\lambda, \mu} D_{p_\lambda}(p_k) D_{p_\mu}(p_k) \left( \frac{p_\lambda * p_\mu}{z_\lambda z_\mu} \right) \\ &= h(t) \sum_{\lambda} D_{p_\lambda}(p_k) D_{p_\lambda}(p_k) \frac{p_\lambda}{z_\lambda} \\ &= h(t) \left( D_{p_{(0)}}(p_k) D_{p_{(0)}}(p_k) p_{(0)} + D_{p_k}(p_k) D_{p_k}(p_k) \frac{p_k}{k} \right) \\ &= h(t)(p_k^2 + kp_k). \end{aligned}$$

Denote by  $\Lambda^*$  the algebra of symmetric power series. The map  $\langle\langle \cdot \rangle\rangle : \Lambda \rightarrow \Lambda^*$  given by  $\langle\langle s_\lambda \rangle\rangle = \sum_n s_{\lambda[n]}$ , is called the *reduced notation for  $s_\lambda$* , which can be extended by linearity, since the Schur functions form a basis of  $\Lambda$ . We have the following characterization for the reduced notation of symmetric functions:

**Proposition 6.1** (Thibon[24]). *For any  $f \in \Lambda$ :*

$$\langle\langle f \rangle\rangle = hD_e(f),$$

where  $D_e = \sum_n (-1)^n D_{e_n}$ .

*Proof.* It suffices to prove this for  $f = s_\lambda$ . Using the same idea from Lemma 4.2:

$$\langle\langle f \rangle\rangle = \sum_n s_{\lambda[n]} = h \sum_k (-1)^k s_{\lambda/1^k}.$$

Since  $s_{\lambda/1^k} = D_{s_{1^k}} s_\lambda$  and  $e_k = s_{1^k}$ :

$$\langle\langle s_\lambda \rangle\rangle = h \sum_k (-1)^k D_{e_k} s_\lambda.$$

And the result follows by linearity. ♣

A reduced notation version of Theorem 6.2 is as follows:

**Theorem 6.6** (Littlewood[13]). *For two fixed partitions  $\lambda, \mu$ :*

$$\langle\langle s_\lambda \rangle\rangle * \langle\langle s_\mu \rangle\rangle = \left\langle \left\langle \sum_{\alpha, \beta, \gamma} s_{\lambda/\alpha\gamma} s_{\mu/\beta\gamma} s_\alpha * s_\beta \right\rangle \right\rangle,$$

where  $\alpha\gamma$  is defined as before (Theorem 6.2).

**Corollary 6.7** (Thibon [24]). *For any  $f, g \in \Lambda$  we have*

$$\langle\langle f \rangle\rangle * \langle\langle g \rangle\rangle = \left\langle \left\langle \sum_{\lambda} (D_{p_{\lambda}} f)(D_{p_{\lambda}} g) \frac{p_{\lambda}[\mathbf{x} + 1]}{z_{\lambda}} \right\rangle \right\rangle.$$

The proof of this Corollary follows from using Proposition 6.1 and setting  $u_{\lambda} = \frac{p_{\lambda}}{z_{\lambda}}$  and  $v_{\lambda} = p_{\lambda}$  in Theorem 6.4.

## 6.4 Relation to D-finiteness

In this section we state our main results relating the previous notions with the theory of D-finite power series.

**Theorem 6.8.** *Let  $f, g$  be polynomials in the  $p_i$ 's and possibly another variable  $t$ . Then  $\langle hf, h(t)g \rangle$  is D-finite as a function of  $t$ .*

*Proof.* Set  $p_i = 1$  for all  $i$  in Theorem 6.5, which gives:

$$\langle hf, h(t)g \rangle = \frac{1}{1-t} (\Phi(t))_{p_i=1} = \frac{1}{1-t} \phi(t),$$

where  $1/(1-t)$  is D-finite and  $\phi(t)$  is a polynomial in  $t$ , because  $f$  and  $g$  are polynomials. The result follows from the basic properties of D-finiteness.  $\clubsuit$

We now present an additional proof to the previous theorem using a property of the adjoint multiplication of  $h$ .

*Additional proof.* From the definition of adjoint multiplication,

$$\langle hf, h(t)g \rangle = \langle f, D_h h(t)g \rangle.$$

Notice now that  $f$  is a polynomial and so it is D-finite in the  $p_i$ 's and  $t$ , and it involves a finite number of  $p_i$ 's. Then it remains for us to show that  $D_h h(t)g$  is D-finite. Indeed:

$$\begin{aligned} D_h h(t)g &= \sum_n t^n D_h(h_n g), \\ &= \sum_n t^n (h_n g)[\mathbf{x} + 1], \quad \text{since } D_h(g) = g[\mathbf{x} + 1], \text{ see [24]} \\ &= \sum_n t^n h_n[\mathbf{x} + 1]g[\mathbf{x} + 1], \end{aligned}$$

where  $g[\mathbf{x} + 1]$  is a D-finite polynomial and

$$\begin{aligned} \sum_n t^n h_n[\mathbf{x} + 1] &= \exp\left(\sum_k \frac{t^k (p_k + 1)}{k}\right), \\ &= \exp\left(\sum_k t^k \frac{p_k}{k}\right) \exp\left(\sum_k \frac{t^k}{k}\right), \\ &= h(t) \frac{1}{1-t}, \text{ which is D-finite.} \end{aligned}$$

By Theorem 5.3 we have that  $\langle hf, h(t)g \rangle$  is D-finite as wanted. ♣

Notice that this proof holds also if  $f$  and  $g$  involve a finite number of  $p_i$ 's but are not necessarily polynomials. In this case  $g[\mathbf{x} + 1]$  results from replacing  $p_i \rightarrow p_i + 1$  in  $g$  for a finite number of  $p_i$ 's, which preserves D-finiteness by Theorem 4.2. In a similar way using that  $D_e(f) = f[\mathbf{x} - 1]$ , we have the following result:

**Proposition 6.2.** *Let  $f, g$  be D-finite in the  $p_i$ 's and another variable  $t$ , such that they both involve a finite number of  $p_i$ 's, and define  $e, h, e(t)h(t), s(t), s(t)e(t)^{-1}$  and  $s(t)h(t)^{-1}$  as in Theorem 5.2. Then the scalar products  $\langle ef, G \rangle$  and  $\langle hf, G \rangle$  with*

$$G \in \{e(t)g, h(t)g, s(t)g, s(t)e(t)^{-1}g, s(t)h(t)^{-1}g\}$$

are all D-finite as functions of  $t$ .

Both Theorem 6.8 and Proposition 6.2 provide large families of examples of D-finite scalar products for which Gessel's conditions are not satisfied. However we would like to answer more general questions in future research:

**Problem 6.2.** *Let  $F, G, f$  and  $g$  be series in the  $p_i$ 's and another variable  $t$  such that  $f$  and  $g$  involve a finite number of  $p_i$ 's, and  $\langle F, G \rangle$  is D-finite in  $t$ . Then is  $\langle Ff, Gg \rangle$  D-finite in  $t$  as well? Which hypotheses can be added so that this statement is true?*

**Problem 6.3.** *Can we characterize precisely the pairs of formal power series  $f(t, p_1, p_2, \dots)$ ,  $g(t, p_1, p_2, \dots)$  whose scalar product is D-finite? In other words, can we weaken the conditions on Theorem 5.3 such that they are sufficient and necessary?*

**Proposition 6.3.** *If  $f$  and  $g$  are two D-finite symmetric functions then  $hf * hg = h\Phi$  is D-finite. Moreover,  $\Phi$  is D-finite.*

*Proof.* Use the closure properties of D-finiteness. Since  $h$  is D-finite, we have that  $hf$  and  $hg$  are D-finite as well. By Theorem 5.1,  $h\Phi$  is D-finite. On the other hand,  $h = \exp\left(\sum_k \frac{p_k}{k}\right)$  and  $h^{-1} = \exp\left(-\sum_k \frac{p_k}{k}\right)$ , which is also D-finite, and so

$$\Phi = \exp\left(-\sum_k \frac{p_k}{k}\right) hf * hg \text{ is D-finite.}$$

♣

The symmetric function  $\Phi$  is known as the *Smash product* of  $f$  and  $g$ . See the work by Marcelo Aguiar [1] for more on this operation.

Some of the original results from this section need Theorem 6.4 for their proof. However they do not require the whole statement, as the case  $u_\lambda = p_\lambda/z_\lambda$ ,  $v_\lambda = p_\lambda$  is sufficient for all of them. In order to avoid dependance on the theory of Hopf algebras, and to illustrate the deep relationship between the theory developed by Thibon, and that of D-finite symmetric functions, we provide below and much simpler original proof of this particular case, making use of Maple and the algorithms [5].

**Theorem 6.9.** *The following relation is always satisfied:*

$$(hp_\lambda) * (hp_\mu) = h \sum_{\gamma} D_{p_\gamma}(p_\lambda) D_{p_\gamma}(p_\mu) \frac{p_\gamma}{z_\gamma}. \quad (6.1)$$

Generally for any two symmetric functions  $f$  and  $g$  we have:

$$(hf) * (hg) = h \sum_{\gamma} D_{p_\gamma}(f) D_{p_\gamma}(g) \frac{p_\gamma}{z_\gamma}. \quad (6.2)$$

*Proof.* Equation 6.2 is a result of Equation 6.1 so we only need to prove equation 6.1. Take  $p_\lambda = p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n}$  and  $p_\mu = p_1^{s_1} p_2^{s_2} \cdots p_m^{s_m}$ . The left hand side can be simplified as follows

$$(hp_\lambda) * (hp_\mu) = \left[ \exp\left(\sum_{i \geq 1} \frac{p_i}{i}\right) p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n} \right] * \left[ \exp\left(\sum_{j \geq 1} \frac{p_j}{j}\right) p_1^{s_1} p_2^{s_2} \cdots p_m^{s_m} \right].$$

Then,

$$\begin{aligned} (hp_\lambda) * (hp_\mu) &= \left[ \prod_{i \geq 1} \exp\left(\frac{p_i}{i}\right) p_i^{r_i} \right] * \left[ \prod_{j \geq 1} \exp\left(\frac{p_j}{j}\right) p_j^{s_j} \right] \\ &= \prod_{i \geq 1} \left( \exp\left(\frac{p_i}{i}\right) p_i^{r_i} \right) * \left( \exp\left(\frac{p_i}{i}\right) p_i^{s_i} \right), \text{ by Lemma 5.1.} \end{aligned}$$



Using Maple, we get that each term of the form  $L = \exp\left(\frac{p_k}{k}\right) p_k^{r_k} * \exp\left(\frac{p_k}{k}\right) p_k^{s_k}$  satisfies the differential equation:

$$(kr_k s_k - p_k)L + kp_k^2 \frac{\partial^2 L}{\partial p_k^2} + (kp_k - p_k^2 - kr_k p_k - ks_k p_k) \left( \frac{\partial L}{\partial p_k} \right) = 0. \quad (6.3)$$

Then it remains to show that the right hand side of 6.1 satisfies these equations as well. After differentiating, this right hand side can be written as:

$$h \sum_{\substack{t_i \\ t_i \leq r_i \\ t_i \leq s_i}} \prod_i \frac{i^{t_i} t_i! (r_i)_{t_i} (s_i)_{t_i}}{t_i!} p_i^{r_i + s_i - t_i},$$

where  $(r_i)_{t_i}$  represents the descending factorial. We proved that for each  $k$ , the expression above satisfies Equation 6.3, for which we simply expand the result in terms of the power sum basis and verify that all the coefficients are 0. We omit the mechanical calculation here. ♣

**Problem 6.4.** *Can we prove the statement of Theorem 6.9 by using these algorithms, or an extended form involving D-finiteness with respect to the general adjoint bases  $\{u_\lambda\}$  and  $\{v_\lambda\}$ ?*

# Chapter 7

## Conclusion

### 7.1 Summary

The following is a summary of the original results obtained in the previous chapters.

**D-finite sums of Schur functions and Tableaux (Corollary 5.6)** The symmetric series  $\sum_m s_{(m^k, \lambda)/\mu}$  is D-finite. Hence, the number of SSYT of type  $(j^n)$  and shape  $(m^k, \lambda)/\mu$  is a P-recursive function of  $n$ , for  $\lambda, j$  and  $k$  fixed. Particularly, the number of SYT of shape  $(m^k, \lambda)/\mu$  is a P-recursive function of  $m$ .

**Some scalar product identities (Table 5.2)** Mishna [16] provided a family of Kronecker products identities by using the theory of D-finite symmetric functions. From this table we were able to produce a family of scalar products identities (Table 5.2). The pairs of symmetric functions corresponding to the first two lines of this table have D-finite scalar products, yet they do not satisfy the second of Gessel's conditions, which suggests that such condition may be extended (see Problem 6.3).

**D-finite scalar products (Theorem 6.8 and Proposition 6.2)** Let  $f, g$  be power series in the  $p_i$ 's and another variable  $t$ , involving only a finite number of  $p_i$ 's. Then  $\langle hf, h(t)g \rangle$  is D-finite as a function of  $t$ . Moreover, the scalar products  $\langle hf, G \rangle$  and  $\langle ef, G \rangle$  are D-finite for  $G = h(t)g, e(t)g, s(t)g, s(t)e(t)^{-1}g, s(t)h(t)^{-1}g$ . This result provides an extension of Gessel's Theorem 5.3 by weakening the conditions for a scalar product to be D-finite. We present a more general result which would imply the D-finiteness of these products (see Problem 6.2).

**D-finiteness of Smash product (Proposition 6.3)** If  $f$  and  $g$  are two D-finite symmetric functions then  $hf * hg = h\phi$  is D-finite. Moreover,  $\phi$  is D-finite.

**New proof of a particular case of Theorem 6.9** For  $\lambda$  and  $\mu$  fixed, the following relation is satisfied:

$$(hp_\lambda) * (hp_\mu) = h \sum_{\gamma} D_{p_\gamma}(p_\lambda) D_{p_\gamma}(p_\mu) \frac{p_\gamma}{z_\gamma}.$$

Generally for any two symmetric functions  $f$  and  $g$  we have:

$$(hf) * (hg) = h \sum_{\gamma} D_{p_\gamma}(f) D_{p_\gamma}(g) \frac{p_\gamma}{z_\gamma}.$$

Our proof of this result uses only the theory developed by Mishna [15] and Chyzak, Mishna and Salvy [5]. In fact it depends almost entirely on their algorithms implemented in Maple. This is our most important result relating the theory of D-finite symmetric functions with Thibon's work and it may suggest an even deeper relation between these areas (see Problem 6.4).

## 7.2 Open problems

The following are some of the open problems that we propose as a possible extension of our current work.

It seems to us, as suggested by our proof of Theorem 6.9, that many of the results obtained by Thibon [24] and Scharf, Thibon and Wybourne [20] can be explained using the theory of D-finite symmetric series. This opens up a door to a new theory on the connections between D-finiteness and these other areas of research.

**Problem 1.** Can we explain the Kronecker product of two-row shape Schur functions (see Scharf, Thibon and Wybourne [20]) using D-finiteness, more specifically by using the algorithms by Mishna [15] and Chyzak, Mishna and Salvy [5]?

**Problem 2.** Can we prove the statement of Theorem 6.9 by using these algorithms, or an extended form involving D-finiteness with respect to the general adjoint bases  $\{u_\lambda\}$  and  $\{v_\lambda\}$ ?

Concerning an extension of Theorem 5.3, we present the following open problems:

**Problem 3.** Let  $F, G, f$  and  $g$  be series in the  $p_i$ 's and another variable  $t$  such that  $f$  and  $g$  involve only a finite number of  $p_i$ 's and  $\langle F, G \rangle$  is D-finite in  $t$ . Then is  $\langle Ff, Gg \rangle$  is D-finite in  $t$  as well? Which hypotheses can be added so that this statement is true?

**Problem 4.** Can we characterize precisely the pairs of formal power series  $f(t, p_1, p_2, \dots)$ ,  $g(t, p_1, p_2, \dots)$  whose scalar product is D-finite? In other words can be weaken the conditions on Theorem 5.3 such that they are sufficient and necessary?

Regarding the D-finiteness of sums of Schur functions, we ask the following questions:

**Problem 5.** Are the symmetric series  $\sum_n s_{n, n-1, n-2, \dots, 1}$  and  $\sum_n s_{n^n}$  D-finite with respect the  $p_i$ 's?

**Problem 6.** What are some sufficient conditions for the sum of all Schur functions indexed by the skew partitions in a given family  $\mathcal{F}$  to be D-finite? Notice that this applies directly to tableaux enumeration (as in Section 5.3) by showing the P-recursiveness with respect to  $n$  of the number of SSYT of type  $j^n$  and shape in  $\mathcal{F}$ , for any fixed  $j$ .

# Bibliography

- [1] Marcelo Aguiar, Ferrer Walter, and Moreira Walter. The smash product of symmetric functions. extended abstract. *Preprint. arXiv:math/0412016v2*, 2004.
- [2] Jason Bandlow. An elementary proof of the hook formula. *Electronic Journal of Combinatorics*, 15(1):Research paper 45, 14, 2008.
- [3] Edward A. Bender and Donald E. Knuth. Enumeration of plane partitions. *Journal of Combinatorial Theory. Series A*, 13:40–54, 1972.
- [4] Emmanuel Briand, Rosa Orellana, and Mercedes Rosas. The stability of the kronecker product of schur functions. *Preprint. arXiv:0907.4652v2*, 2009.
- [5] Frédéric Chyzak, Marni Mishna, and Bruno Salvy. Effective scalar products of  $D$ -finite symmetric functions. *Journal of Combinatorial Theory. Series A*, 112(1):1–43, 2005.
- [6] Philippe Flajolet and Robert Sedgewick. *Analytic combinatorics*. Cambridge University Press, Cambridge, 2009.
- [7] Ira M. Gessel. Symmetric functions and  $P$ -recursiveness. *Journal of Combinatorial Theory. Series A*, 53(2):257–285, 1990.
- [8] Basil Gordon. Notes on plane partitions. V. *Journal of Combinatorial Theory. Series B*, 11:157–168, 1971.
- [9] Basil Gordon and Lorne Houten. Notes on plane partitions. I, II. *Journal of Combinatorial Theory*, 4:72–80; 81–99, 1968.
- [10] Ian P. Goulden and David M. Jackson. Labelled graphs with small vertex degrees and  $P$ -recursiveness. *Society for Industrial and Applied Mathematics. Journal on Algebraic and Discrete Methods*, 7(1):60–66, 1986.
- [11] Ian P. Goulden and David M. Jackson. *Combinatorial enumeration*. Dover Publications Inc., Mineola, NY, 2004. With a foreword by Gian-Carlo Rota, Reprint of the 1983 original.
- [12] Leonard Lipshitz. The diagonal of a  $D$ -finite power series is  $D$ -finite. *Journal of Algebra*, 113(2):373–378, 1988.

- [13] D. E. Littlewood. Products and plethysms of characters with orthogonal, symplectic and symmetric groups. *Canadian Journal of Mathematics*, 10:17–32, 1958.
- [14] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press Oxford University Press, New York, second edition, 1995. With contributions by A. Zelevinsky, Oxford Science Publications.
- [15] Marni Mishna. *A holonomic systems approach to algebraic combinatorics*. PhD thesis, Université du Québec a Montréal, 2003.
- [16] Marni Mishna. Kronecker product identities from D-finite symmetric functions. *Séminaire Lotharingien de Combinatoire*, 57:Art. B57c, 7, 2006/08.
- [17] Francis D. Murnaghan. The Analysis of the Kronecker Product of Irreducible Representations of the Symmetric Group. *American Journal of Mathematics*, 60(3):761–784, 1938.
- [18] Francis D. Murnaghan. On the analysis of the Kronecker product of irreducible representations of  $S_n$ . *Proceedings of the National Academy of Sciences of the United States of America*, 41:515–518, 1955.
- [19] Howard J. Redfield. The Theory of Group-Reduced Distributions. *American Journal of Mathematics*, 49(3):433–455, 1927.
- [20] Thomas Scharf, Jean-Yves Thibon, and Brian G. Wybourne. Reduced notation, inner plethysms and the symmetric group. *Journal of Physics. A. Mathematical and General*, 26(24):7461–7478, 1993.
- [21] Richard P. Stanley. Differentiably finite power series. *European Journal of Combinatorics*, 1(2):175–188, 1980.
- [22] Richard P. Stanley. *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.
- [23] John Stembridge. Sf package (maple) <http://www.math.lsa.umich.edu/jrs/maple.html>.
- [24] Jean-Yves Thibon. Hopf algebras of symmetric functions and tensor products of symmetric group representations. *International Journal of Algebra and Computation*, 1(2):207–221, 1991.
- [25] Doron Zeilberger. Sister Celine’s technique and its generalizations. *Journal of Mathematical Analysis and Applications*, 85(1):114–145, 1982.